Tutorial:

Multi-Stage Robust Optimization

Wolfram Wiesemann
Department of Analytics, Marketing and Operations (Business School)
Department of Computing (by courtesy)
Imperial College London, UK

Agenda

Part 1 Single- vs Multi-Stage RO, Complexity

Part 2 Continuous Recourse Decisions

- Two-Stage Models

Part 3 Continuous Recourse Decisions

- Multi-Stage Models

Part 4 Discrete Recourse Decisions

Part 5 Future Research Directions

Deterministic optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x, \xi) & \\ \text{subject to} & f_i(x, \xi) \leq 0 & \forall i = 1, \ldots, m \end{array}$$

Deterministic optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x,\xi) & \\ \text{subject to} & f_i(x,\xi) \leq 0 & \forall i = 1,\ldots,m \end{array}$$

Stochastic optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \mathbb{E}_{\mathbb{P}}\left[f_0(x,\tilde{\xi})\right] & \\ \text{subject to} & f_i(x,\tilde{\xi}) \leq 0 & \mathbb{P}\text{-a.s. } \forall i = 1,\ldots,m \end{array}$$

Deterministic optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x, \xi) & \\ \text{subject to} & f_i(x, \xi) \leq 0 & \forall i = 1, \ldots, m \end{array}$$

Stochastic optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \mathbb{E}_{\mathbb{P}}\left[f_0(x, \tilde{\xi})\right] & \\ \text{subject to} & f_i(x, \tilde{\xi}) \leq 0 & \mathbb{P}\text{-a.s. } \forall i = 1, \ldots, m \end{array}$$

Robust optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \left[\max_{\xi \in \Xi} f_0(x, \xi)\right] & \\ \text{subject to} & f_i(x, \xi) \leq 0 & \forall \xi \in \Xi, \forall i = 1, \ldots, m \end{array}$$

Deterministic optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x, \xi) & \\ \text{subject to} & f_i(x, \xi) \leq 0 & \forall i = 1, \ldots, m \end{array}$$

Distributionally robust optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \left[\sup\limits_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}_{\mathbb{P}} \left[f_0(x, \tilde{\xi}) \right] \right] & \\ \text{subject to} & f_i(x, \tilde{\xi}) \leq 0 & \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathfrak{P}, \forall i = 1, \ldots, m \end{array}$$

Robust optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \left[\max\limits_{\xi \in \Xi} f_0(x, \xi) \right] & \\ \text{subject to} & f_i(x, \xi) \leq 0 & \forall \xi \in \Xi, \forall i = 1, \ldots, m \end{array}$$

Robust optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \left[\max_{\xi \in \Xi} f_0(x, \xi) \right] & \\ \text{subject to} & f_i(x, \xi) \leq 0 & \forall \xi \in \Xi, \forall i = 1, \ldots, m \end{array}$$

Linear robust optimization:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \left[\max_{\xi \in \Xi} c(\xi)^\top x \right] & \\ \text{subject to} & a_i(\xi)^\top x \leq b_i(\xi) & \forall \xi \in \Xi, \forall i = 1, \ldots, m \end{array}$$

with
$$c(\xi) = C\xi + c$$
$$a_i(\xi) = A_i\xi + a$$
$$b_i(\xi) = b_i^\top \xi + b_i^0$$

Linear robust optimization:

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & \left[\max_{\xi \in \Xi} \; c(\xi)^\top x \right] \\
\text{subject to} \quad & a_i(\xi)^\top x \leq b_i(\xi) \qquad \forall \xi \in \Xi, \forall i = 1, \ldots, m
\end{aligned}
$$

$$
\begin{aligned}
c(\xi) &= C\xi + c \\
a_i(\xi) &= A_i \xi + a \\
b_i(\xi) &= b_i^\top \xi + b_i^0
\end{aligned}
$$

Linear robust optimization:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \left[\max_{\xi \in \Xi} \; c(\xi)^\top x \right] \\ \text{subject to} \quad & a_i(\xi)^\top x \le b_i(\xi) \qquad \forall \xi \in \Xi, \forall i = 1, \dots, m \end{aligned}$$

$$\begin{aligned} c(\xi) &= C\xi + c \\ a_i(\xi) &= A_i\xi + a \\ b_i(\xi) &= b_i^\top \xi + b_i^0 \end{aligned}$$

"Epigraph" trick:

$$\begin{aligned} \underset{x,\tau}{\text{minimize}} \quad & \tau \\ \text{subject to} \quad & \tau = \left[\max_{\xi \in \Xi} \; c(\xi)^\top x \right] \\ & a_i(\xi)^\top x \le b_i(\xi) \qquad \forall \xi \in \Xi, \forall i = 1, \dots, m \end{aligned}$$

Linear robust optimization:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \left[\max_{\xi \in \Xi} \; c(\xi)^\top x\right] \\ \text{subject to} & a_i(\xi)^\top x \le b_i(\xi) \qquad \forall \xi \in \Xi, \forall i = 1,\dots,m \end{array}$$

$$c(\xi) = C\xi + c$$
$$a_i(\xi) = A_i\xi + a$$
$$b_i(\xi) = b_i^\top \xi + b_i^0$$

"Epigraph" trick:

$$\begin{array}{ll} \underset{x,\tau}{\text{minimize}} & \tau \\ \text{subject to} & \tau \ge \left[\max_{\xi \in \Xi} \; c(\xi)^\top x\right] \\ & a_i(\xi)^\top x \le b_i(\xi) \qquad \forall \xi \in \Xi, \forall i = 1,\dots,m \end{array}$$

Linear robust optimization:

$$\underset{x}{\text{minimize}} \quad \left[\max_{\xi \in \Xi} \; c(\xi)^\top x \right]$$
$$\text{subject to} \quad a_i(\xi)^\top x \le b_i(\xi) \qquad \forall \xi \in \Xi, \forall i = 1, \ldots, m$$

$$c(\xi) = C\xi + c$$
$$a_i(\xi) = A_i \xi + a$$
$$b_i(\xi) = b_i^\top \xi + b_i^0$$

"Epigraph" trick:

$$\underset{x,\tau}{\text{minimize}} \quad \tau$$
$$\text{subject to} \quad \tau \ge c(\xi)^\top x \qquad \forall \xi \in \Xi$$
$$a_i(\xi)^\top x \le b_i(\xi) \qquad \forall \xi \in \Xi, \forall i = 1, \ldots, m$$

Semi-infinite constraint:

$$a_i(\xi)^\top x \leq b_i(\xi) \qquad \forall \xi \in \Xi$$

$$a_i(\xi) = A_i\xi + a$$

$$b_i(\xi) = b_i^\top \xi + b_i^0$$

Semi-infinite constraint:

$$a_i(\xi)^\top x \leq b_i(\xi) \qquad \forall \xi \in \Xi$$

$$a_i(\xi) = A_i \xi + a$$

$$b_i(\xi) = b_i^\top \xi + b_i^0$$

$$\Xi = \{\xi \in \mathbb{R}^k \, : \, F\xi \geq g\}$$

Semi-infinite constraint:

$$a_i(\xi)^\top x \leq b_i(\xi) \quad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$a_i(\xi)^\top x - b_i(\xi) \leq 0 \quad \forall \xi \in \Xi$$

$$a_i(\xi) = A_i\xi + a$$

$$b_i(\xi) = b_i^\top \xi + b_i^0$$

$$\Xi = \{\xi \in \mathbb{R}^k \,:\, F\xi \geq g\}$$

Semi-infinite constraint:

$$a_i(\xi)^\top x \le b_i(\xi) \quad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$a_i(\xi)^\top x - b_i(\xi) \le 0 \quad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$\max_{\xi \in \Xi} \left[a_i(\xi)^\top x - b_i(\xi) \right] \le 0$$

$$a_i(\xi) = A_i \xi + a$$

$$b_i(\xi) = b_i^\top \xi + b_i^0$$

$$\Xi = \{\xi \in \mathbb{R}^k \,:\, F\xi \ge g\}$$

Semi-infinite constraint:

$$a_i(\xi)^\top x \leq b_i(\xi) \qquad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$a_i(\xi)^\top x - b_i(\xi) \leq 0 \quad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$\max_{\xi \in \Xi} \left[a_i(\xi)^\top x - b_i(\xi) \right] \leq 0 \quad \Longleftrightarrow \quad \begin{array}{ll} \underset{\xi}{\text{maximize}} & a_i(\xi)^\top x - b_i(\xi) \\ \text{subject to} & \xi \in \Xi \end{array} \quad \leq 0$$

$$a_i(\xi) = A_i \xi + a$$

$$b_i(\xi) = b_i^\top \xi + b_i^0$$

$$\Xi = \{\xi \in \mathbb{R}^k \,:\, F\xi \geq g\}$$

Semi-infinite constraint:

$$a_i(\xi)^\top x \le b_i(\xi) \qquad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$a_i(\xi)^\top x - b_i(\xi) \le 0 \quad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$\max_{\xi \in \Xi} \left[a_i(\xi)^\top x - b_i(\xi) \right] \le 0 \quad \Longleftrightarrow \quad \begin{array}{ll} \underset{\xi}{\text{maximize}} & [A_i\xi + a]^\top x - [b_i^\top \xi + b_i^0] \\ \text{subject to} & F\xi \ge g \end{array} \le 0$$

$$a_i(\xi) = A_i\xi + a$$

$$b_i(\xi) = b_i^\top \xi + b_i^0$$

$$\Xi = \{\xi \in \mathbb{R}^k \,:\, F\xi \ge g\}$$

Semi-infinite constraint:

$$a_i(\xi)^\top x \leq b_i(\xi) \qquad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$a_i(\xi)^\top x - b_i(\xi) \leq 0 \quad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$\max_{\xi \in \Xi} \left[a_i(\xi)^\top x - b_i(\xi)\right] \leq 0 \quad \Longleftrightarrow \quad \begin{array}{ll} \underset{\xi}{\text{maximize}} & \left[A_i^\top x - b_i\right]^\top \xi + \left[a^\top x - b_i^0\right] \\ \text{subject to} & -F\xi \leq -g \end{array} \leq 0$$

$$a_i(\xi) = A_i \xi + a$$

$$b_i(\xi) = b_i^\top \xi + b_i^0$$

$$\Xi = \{\xi \in \mathbb{R}^k \,:\, F\xi \geq g\}$$

Semi-infinite constraint:

$$a_i(\xi)^\top x \leq b_i(\xi) \qquad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$a_i(\xi)^\top x - b_i(\xi) \leq 0 \quad \forall \xi \in \Xi$$

$$\Updownarrow$$

$$\max_{\xi \in \Xi} \left[a_i(\xi)^\top x - b_i(\xi)\right] \leq 0$$

$$a_i(\xi) = A_i \xi + a$$

$$b_i(\xi) = b_i^\top \xi + b_i^0$$

$$\Xi = \{\xi \in \mathbb{R}^k \,:\, F\xi \geq g\}$$

$$\Longleftrightarrow$$

$$\begin{array}{ll} \underset{\xi}{\text{maximize}} & \left[A_i^\top x - b_i\right]^\top \xi + \left[a^\top x - b_i^0\right] \\ \text{subject to} & -F\xi \leq -g \end{array} \quad \leq 0$$

$$\Updownarrow$$

$$\begin{array}{ll} \underset{\lambda \geq 0}{\text{minimize}} & -g^\top \lambda + \left[a^\top x - b_i^0\right] \\ \text{subject to} & -F^\top = \left[A_i^\top x - b_i\right] \end{array} \quad \leq 0$$

Single-Stage Robust Optimization

Semi-infinite constraint:

$$a_i(\xi)^\top x \leq b_i(\xi) \qquad \forall \xi \in \Xi$$

$$a_i(\xi) = A_i \xi + a$$
$$b_i(\xi) = b_i^\top \xi + b_i^0$$
$$\Xi = \{\xi \in \mathbb{R}^k \, : \, F\xi \geq g\}$$

$$\Updownarrow$$

$$a_i(\xi)^\top x - b_i(\xi) \leq 0 \quad \forall \xi \in \Xi$$

$$\Updownarrow$$

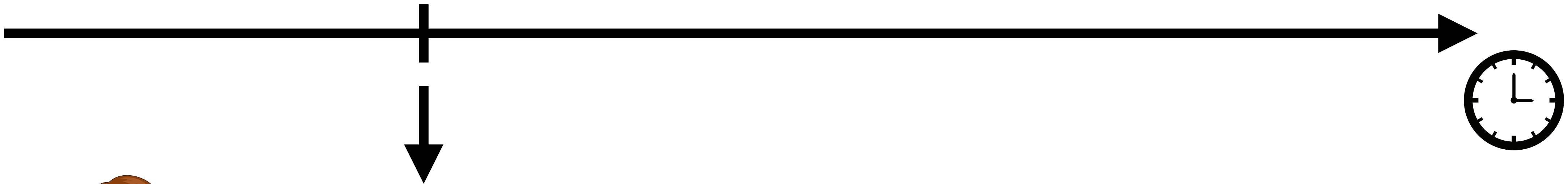
$$\max_{\xi \in \Xi} \left[a_i(\xi)^\top x - b_i(\xi)\right] \leq 0 \quad \Longleftrightarrow \quad \begin{array}{ll} \underset{\xi}{\text{maximize}} & \left[A_i^\top x - b_i\right]^\top \xi + \left[a^\top x - b_i^0\right] \\ \text{subject to} & -F\xi \leq -g \end{array} \quad \leq 0$$

$$\Updownarrow$$

$$\begin{array}{l} -g^\top \lambda + \left[a^\top x - b_i^0\right] \leq 0 \\ -F^\top = \left[A_i^\top x - b_i\right], \;\; \lambda \geq 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \underset{\lambda \geq 0}{\text{minimize}} & -g^\top \lambda + \left[a^\top x - b_i^0\right] \\ \text{subject to} & -F^\top = \left[A_i^\top x - b_i\right] \end{array} \quad \leq 0$$

Single-Stage vs. Multi-Stage Robust Optimization

Single-stage robust optimization:



$$
\begin{aligned}
\min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{x} \in \mathcal{X}
\end{aligned}
$$

Single-Stage vs. Multi-Stage Robust Optimization

Single-stage robust optimization:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi})$$

Single-Stage vs. Multi-Stage Robust Optimization

Single-stage robust optimization:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi})$$

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max\limits_{\boldsymbol{\xi} \in \Xi} \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X} \end{array}$$

Single-Stage vs. Multi-Stage Robust Optimization

Single-stage robust optimization:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi})$$

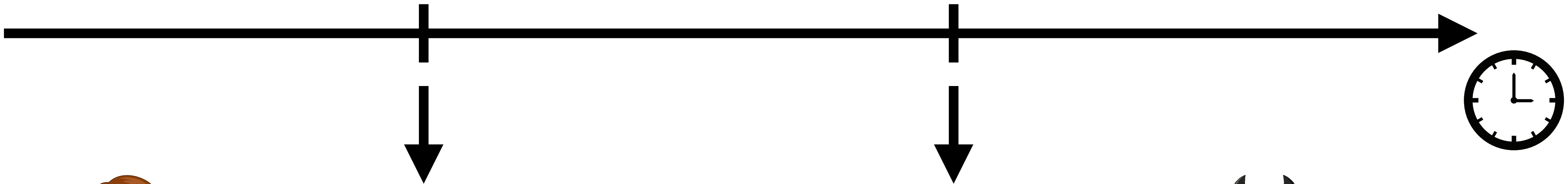
$$\begin{array}{lll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max\limits_{\boldsymbol{\xi} \in \Xi} \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X} & \end{array}$$

$$\begin{array}{lll} \underset{\boldsymbol{x},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X} & \end{array}$$

epigraph reformulation

Single-Stage vs. Multi-Stage Robust Optimization

Single-stage robust optimization:



$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi})$$

$$\begin{array}{lll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max_{\boldsymbol{\xi} \in \Xi} \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X} & \end{array}$$

$$\begin{array}{lll} \underset{\boldsymbol{x},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X} & \end{array}$$

epigraph reformulation

robust optimization trick

Single-Stage vs. Multi-Stage Robust Optimization

Two-stage robust optimization:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\cancel{\boldsymbol{V}(\boldsymbol{\xi})\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi})}$$

Single-Stage vs. Multi-Stage Robust Optimization

Two-stage robust optimization:

$$\begin{array}{ll}\min & \boldsymbol{c}^\top \boldsymbol{x}\\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X}\end{array} \quad + \quad \begin{array}{ll}\max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}\\ \text{s.t.} & \boldsymbol{\xi} \in \Xi\end{array} \quad + \quad \begin{array}{ll}\min & \boldsymbol{r}^\top \boldsymbol{y}\\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y}\end{array}$$

$$\cancel{\boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi})} \Rightarrow \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

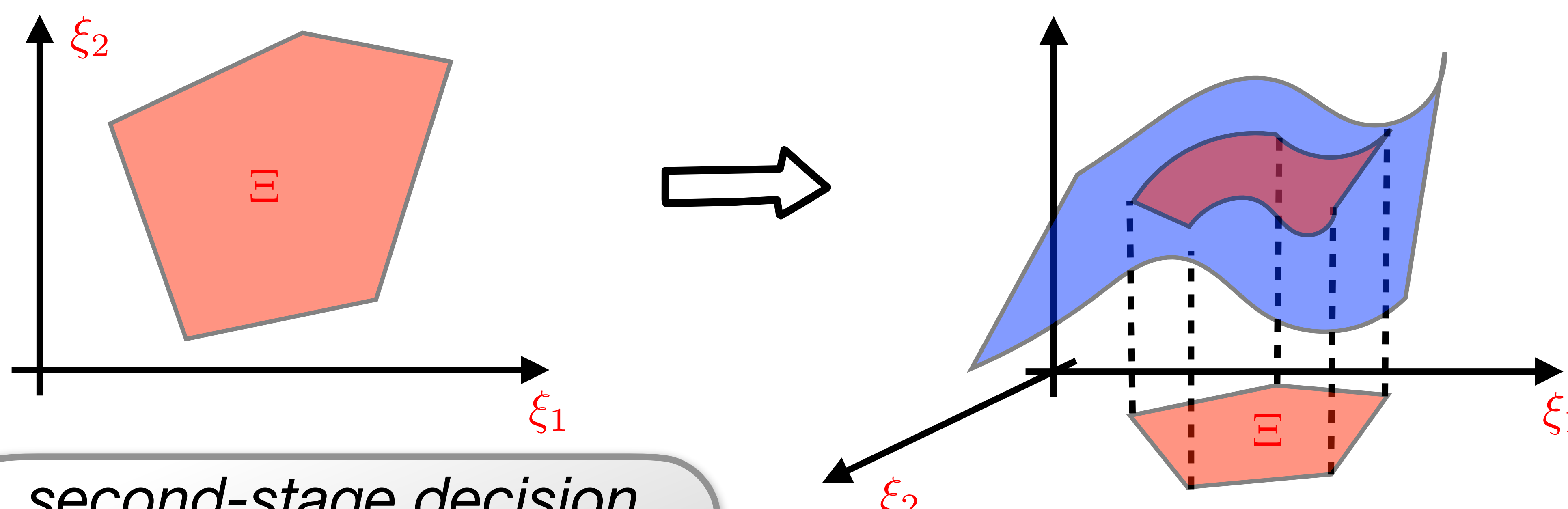


Single-Stage vs. Multi-Stage Robust Optimization

Two-stage robust optimization:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\cancel{\boldsymbol{V}(\boldsymbol{\xi})\,\boldsymbol{x} \geq \boldsymbol{v}(\boldsymbol{\xi})} \;\Longrightarrow\; \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$



second-stage decision depends on uncertainty!

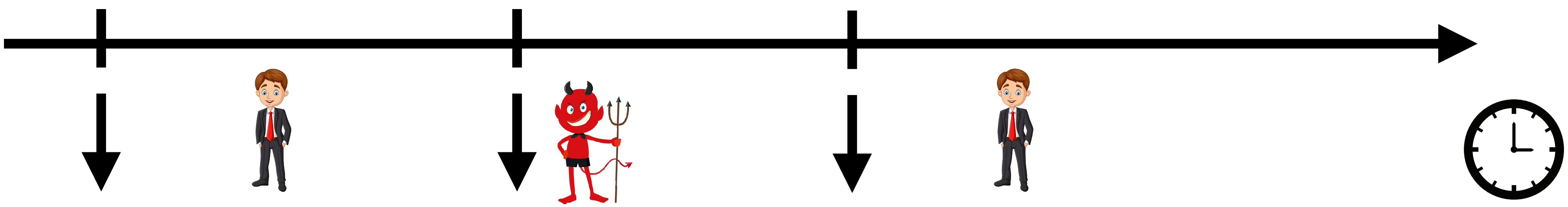
Single-Stage vs. Multi-Stage Robust Optimization

Multi-stage robust optimization:

$$\begin{array}{ll} \min & \boldsymbol{q}_1^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1 \end{array}$$

Single-Stage vs. Multi-Stage Robust Optimization

Multi-stage robust optimization:

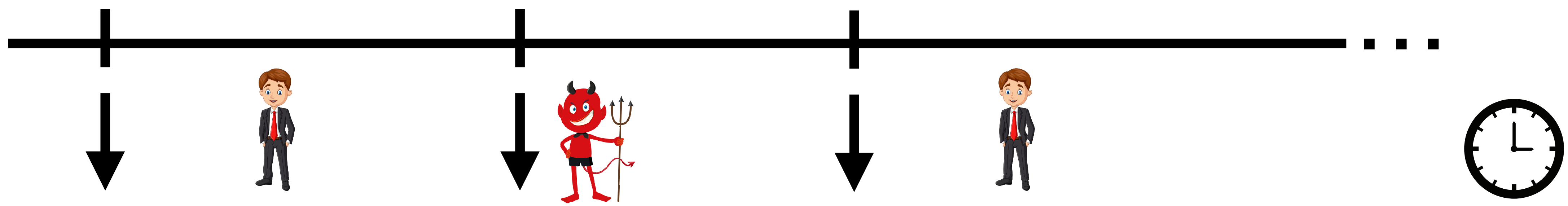


$$\begin{array}{ll}\min & \boldsymbol{q}_1{}^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1\end{array} \; + \; \max_{\boldsymbol{\xi}_2 \in \Xi_2} \; \begin{array}{ll}\min & \boldsymbol{q}_2{}^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{x}_2 \in \mathcal{X}_2\end{array}$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}^2)\,\boldsymbol{x}_1 + \boldsymbol{W}_2\boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}^2)$$

Single-Stage vs. Multi-Stage Robust Optimization

Multi-stage robust optimization:



$$\begin{array}{ll} \min & \boldsymbol{q}_1^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1 \end{array} + \max_{\boldsymbol{\xi}_2 \in \Xi_2} \begin{array}{ll} \min & \boldsymbol{q}_2^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{x}_2 \in \mathcal{X}_2 \end{array} +$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}^2)\,\boldsymbol{x}_1 + \boldsymbol{W}_2 \boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}^2)$$

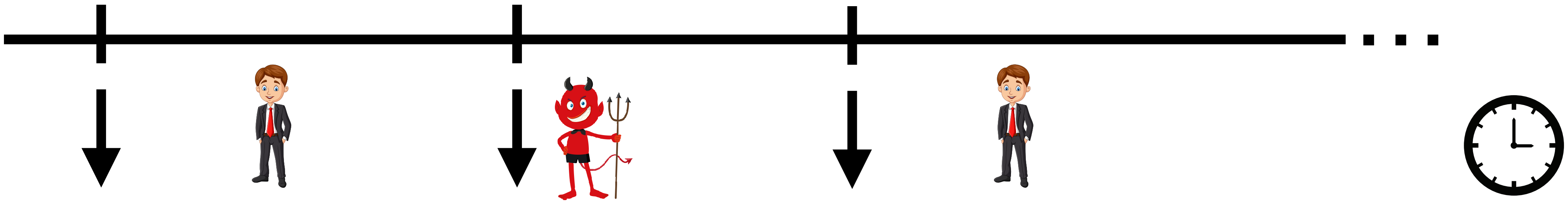
$$\boldsymbol{\xi}^t = \begin{pmatrix} \boldsymbol{\xi}_2 \\ \vdots \\ \boldsymbol{\xi}_t \end{pmatrix}$$

$$\max_{\boldsymbol{\xi}_3 \in \Xi_3(\boldsymbol{\xi}^2)} \begin{array}{ll} \min & \boldsymbol{q}_3^\top \boldsymbol{x}_3 \\ \text{s.t.} & \boldsymbol{x}_3 \in \mathcal{X}_3 \end{array}$$

$$\boldsymbol{T}_3(\boldsymbol{\xi}^3)\,\boldsymbol{x}_2 + \boldsymbol{W}_3 \boldsymbol{x}_3 \geq \boldsymbol{h}_3(\boldsymbol{\xi}^3)$$

Single-Stage vs. Multi-Stage Robust Optimization

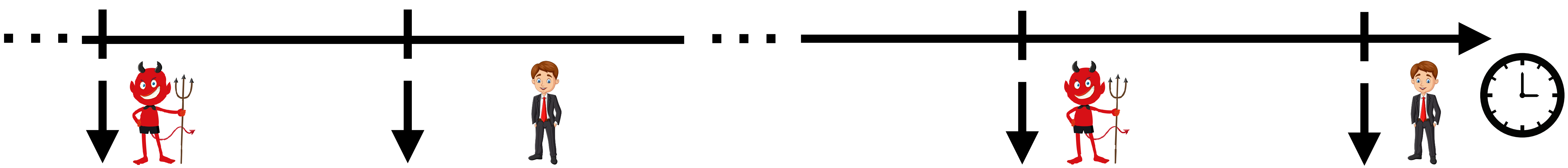
Multi-stage robust optimization:



$$\begin{array}{ll} \min & \boldsymbol{q}_1{}^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1 \end{array} \; + \; \max_{\boldsymbol{\xi}_2 \in \Xi_2} \; \begin{array}{ll} \min & \boldsymbol{q}_2{}^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{x}_2 \in \mathcal{X}_2 \end{array} \; +$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}^2)\, \boldsymbol{x}_1 + \boldsymbol{W}_2 \boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}^2)$$

$$\boldsymbol{\xi}^t = \begin{pmatrix} \boldsymbol{\xi}_2 \\ \vdots \\ \boldsymbol{\xi}_t \end{pmatrix}$$



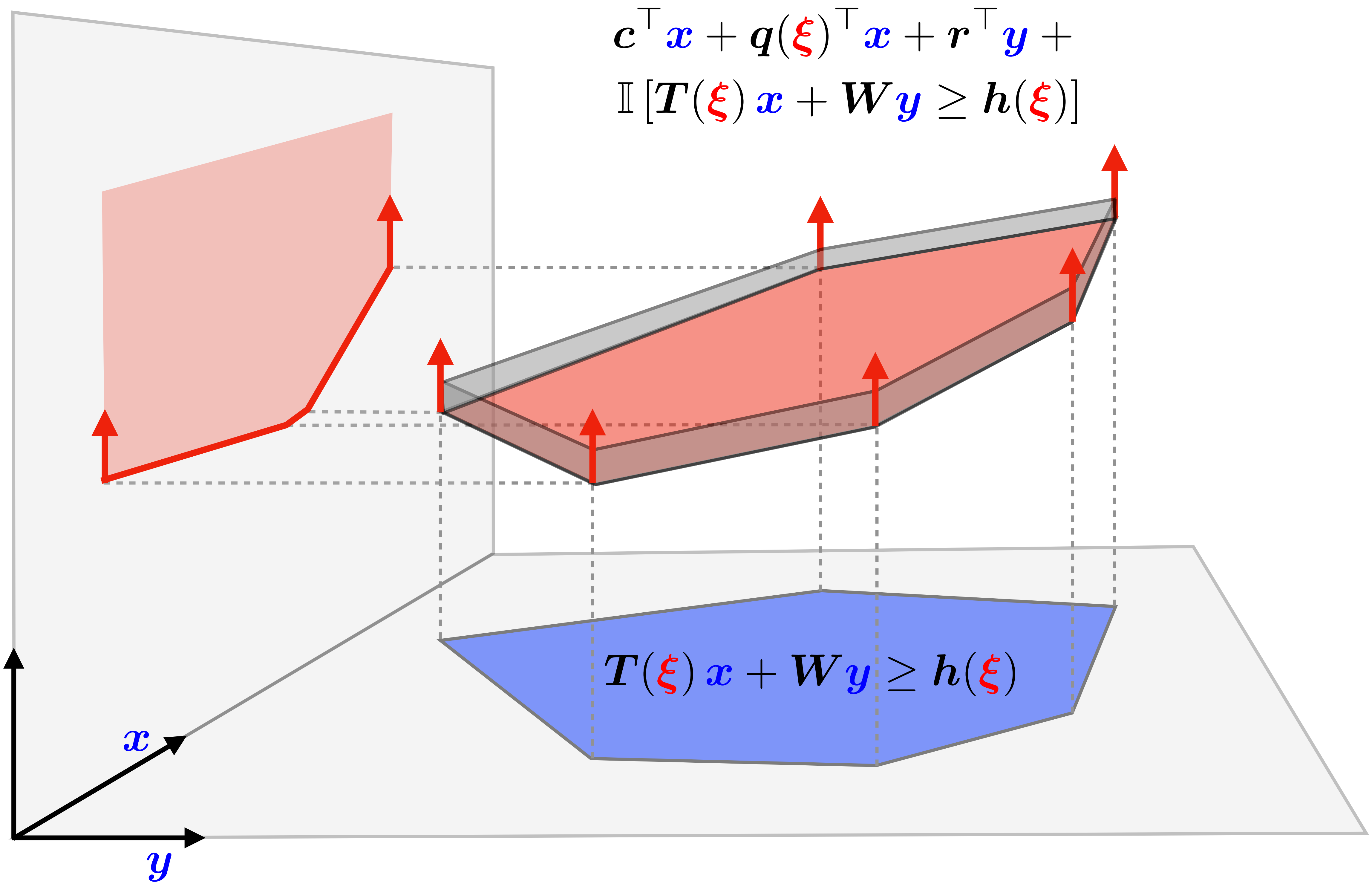
$$\max_{\boldsymbol{\xi}_3 \in \Xi_3(\boldsymbol{\xi}^2)} \; \begin{array}{ll} \min & \boldsymbol{q}_3{}^\top \boldsymbol{x}_3 \\ \text{s.t.} & \boldsymbol{x}_3 \in \mathcal{X}_3 \end{array} \; + \cdots + \; \max_{\boldsymbol{\xi}_T \in \Xi_T(\boldsymbol{\xi}^{T-1})} \; \begin{array}{ll} \min & \boldsymbol{q}_T{}^\top \boldsymbol{x}_T \\ \text{s.t.} & \boldsymbol{x}_T \in \mathcal{X}_T \end{array}$$

$$\boldsymbol{T}_3(\boldsymbol{\xi}^3)\, \boldsymbol{x}_2 + \boldsymbol{W}_3 \boldsymbol{x}_3 \geq \boldsymbol{h}_3(\boldsymbol{\xi}^3)$$

$$\boldsymbol{T}_T(\boldsymbol{\xi}^T)\, \boldsymbol{x}_{T-1} + \boldsymbol{W}_T \boldsymbol{x}_T \geq \boldsymbol{h}_T(\boldsymbol{\xi}^T)$$

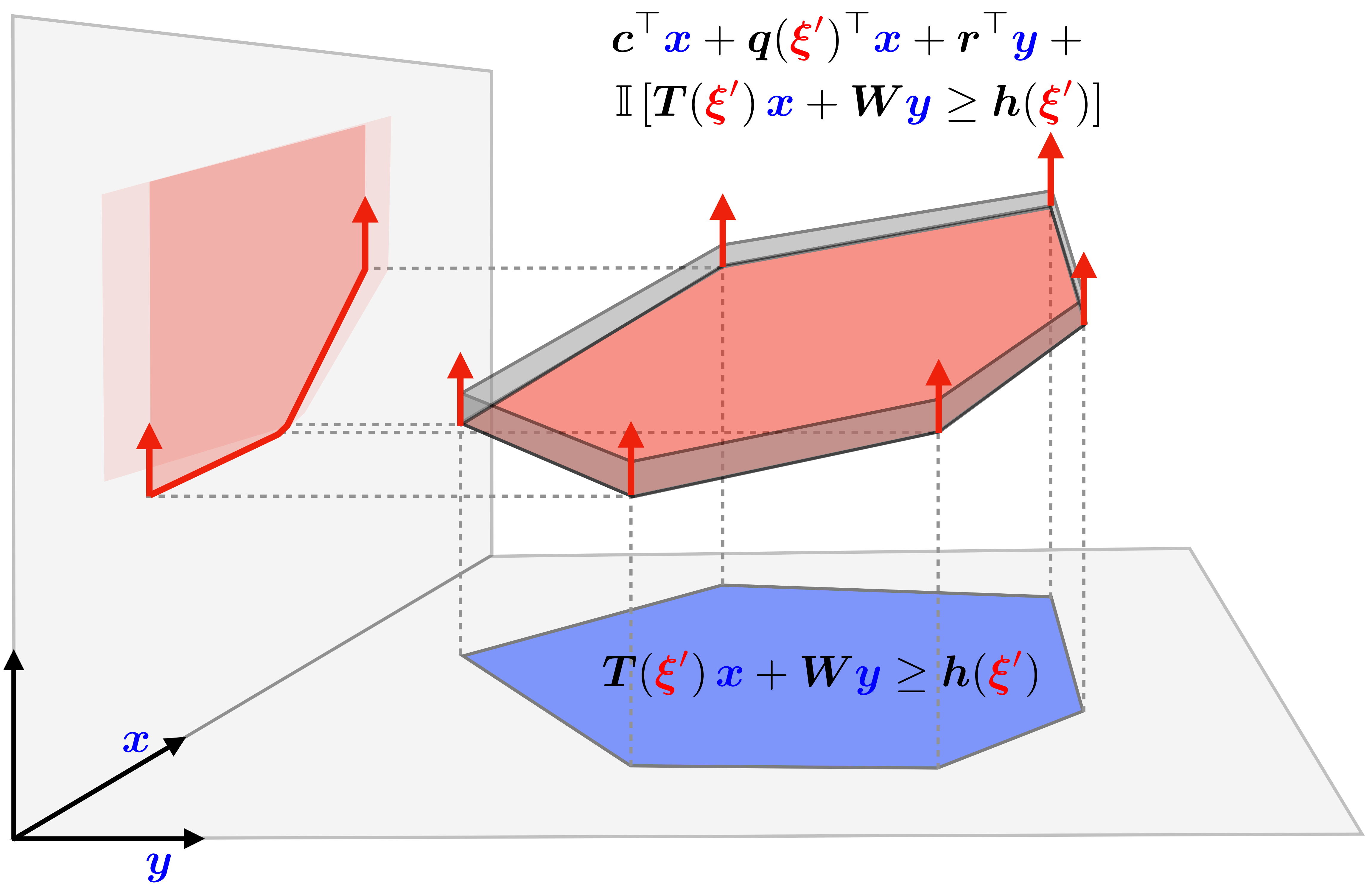
Is Two-Stage Robust Optimization Difficult?

Joint first/second-stage feasible region for fixed $\xi \in \Xi$:



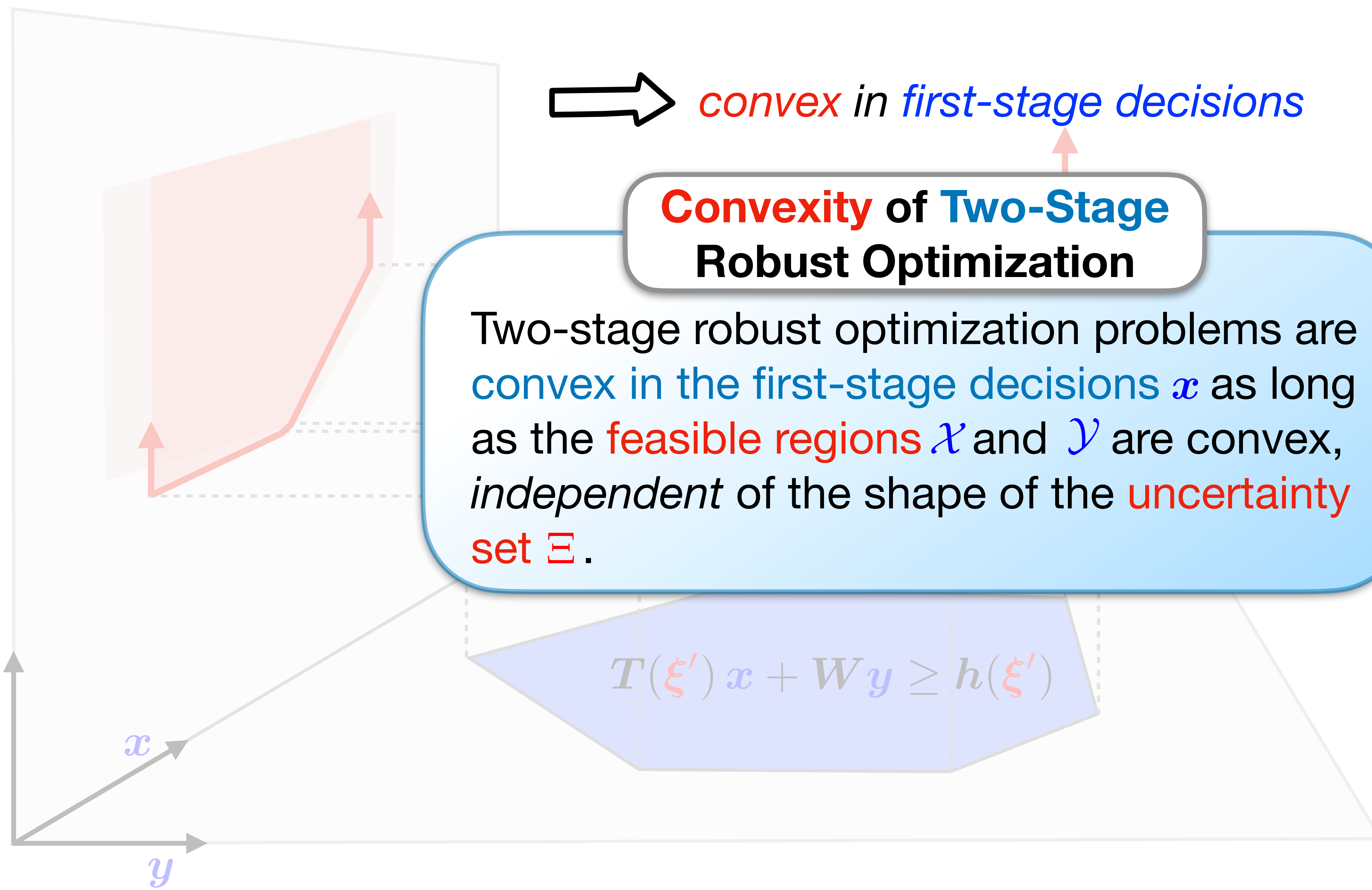
Is Two-Stage Robust Optimization Difficult?

Joint first/second-stage feasible region for fixed $\xi \in \Xi$:



Is Two-Stage Robust Optimization Difficult?

Joint first/second-stage feasible region for fixed $\xi \in \Xi$:



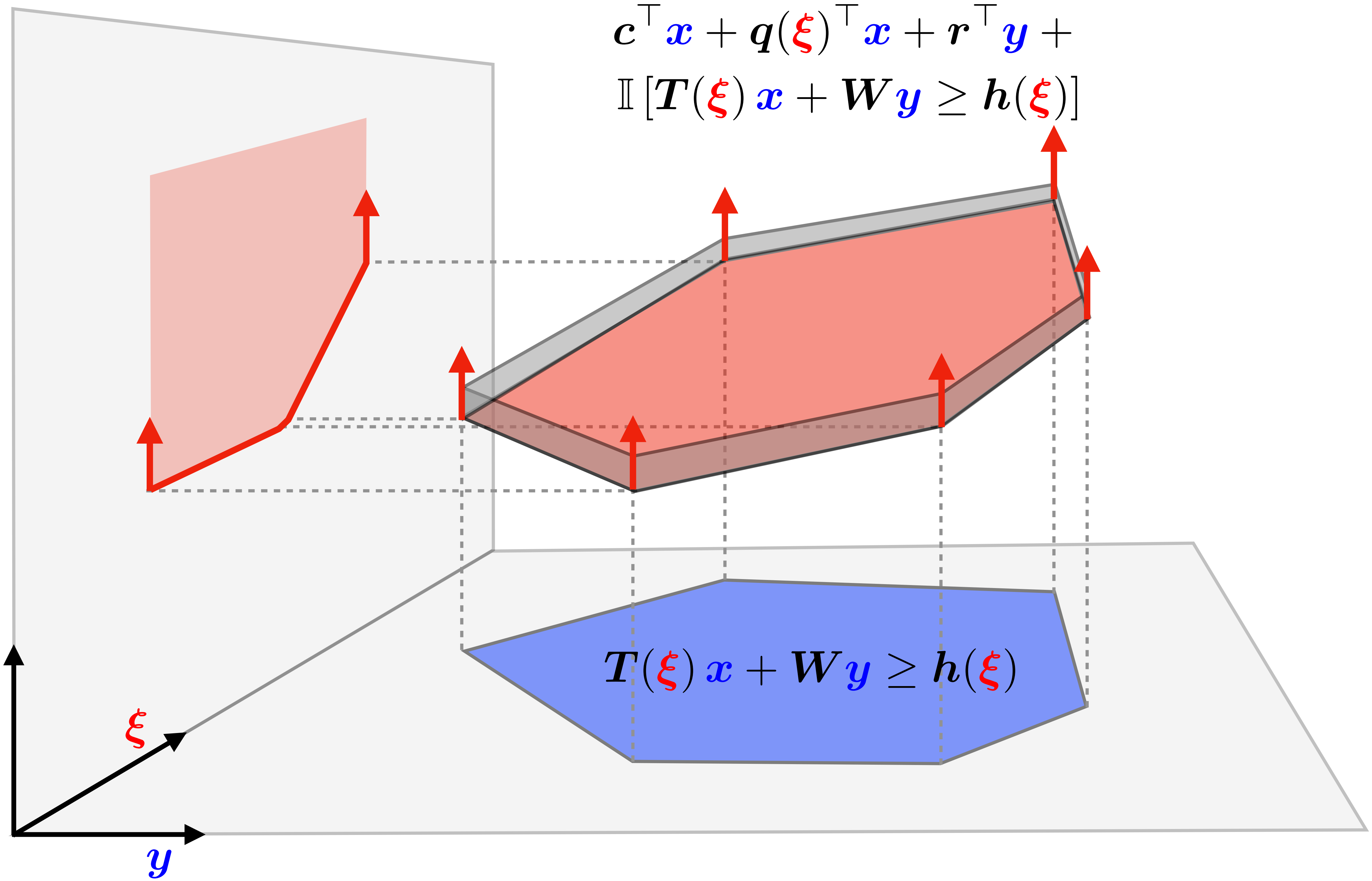
⇨ *convex in first-stage decisions*

Convexity of Two-Stage Robust Optimization

Two-stage robust optimization problems are convex in the first-stage decisions x as long as the feasible regions $\mathcal{X}$ and $\mathcal{Y}$ are convex, *independent* of the shape of the uncertainty set Ξ.

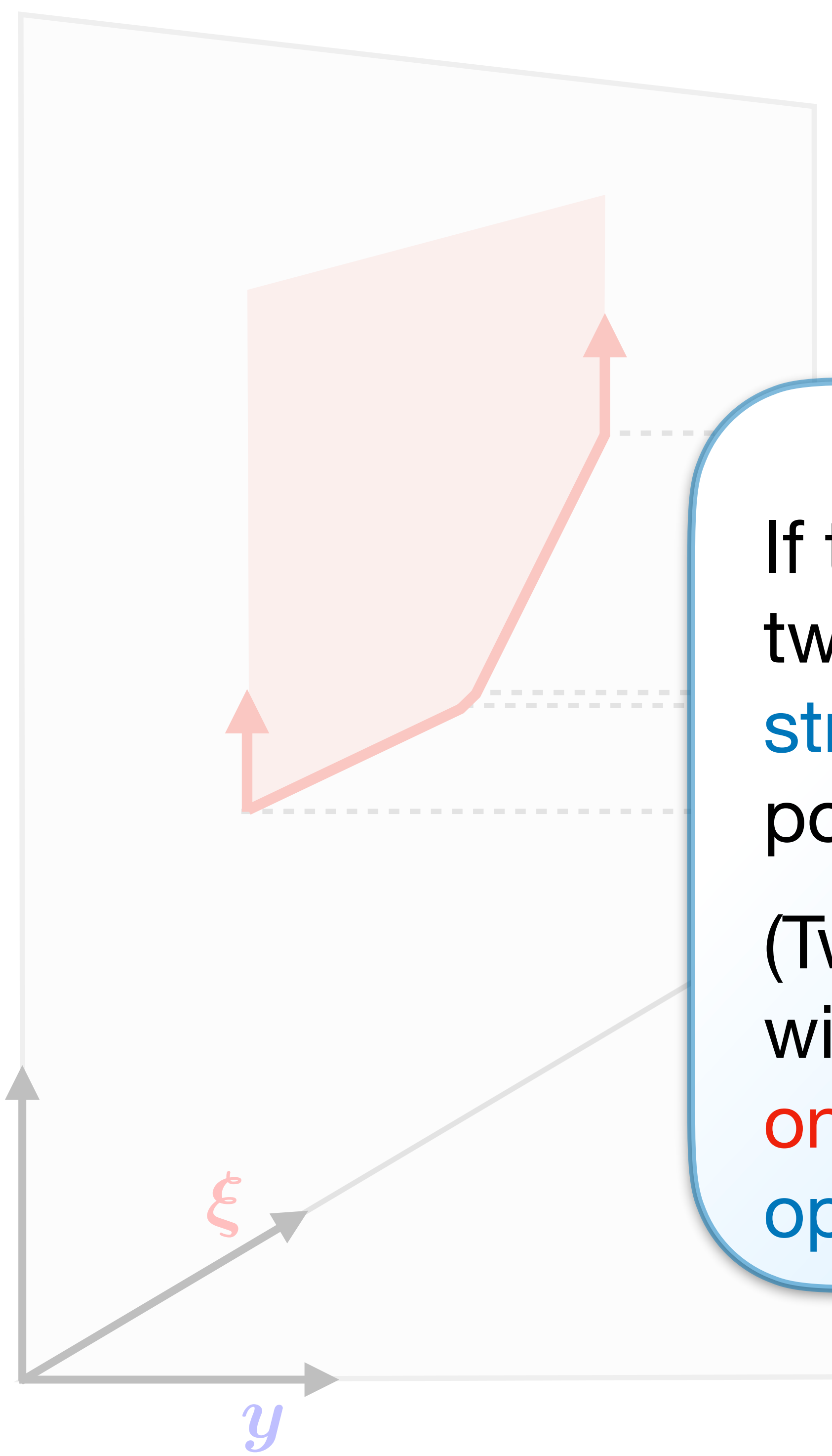
Is Two-Stage Robust Optimization Difficult?

Feasible region for fixed $\boldsymbol{x} \in \mathcal{X}$:



Is Two-Stage Robust Optimization Difficult?

Feasible region for fixed $x \in \mathcal{X}$:



$\Rightarrow$ *convex in uncertainty*

Complexity of Two-Stage Robust Optimization

If the constraints are subject to uncertainty, two-stage robust optimization problems are strongly NP-hard even if all involved sets are polyhedral.

(Two-stage robust optimization problems with uncertainty in the objective function only reduce to single-stage robust optimization problems.)

Agenda

~~Part 1~~ ~~Single- and Multi-Stage RO, Complexity~~

Part 2 Continuous Recourse Decisions

- Two-Stage Models

Part 3 Continuous Recourse Decisions

- Multi-Stage Models

Part 4 Discrete Recourse Decisions

Part 5 Future Research Directions

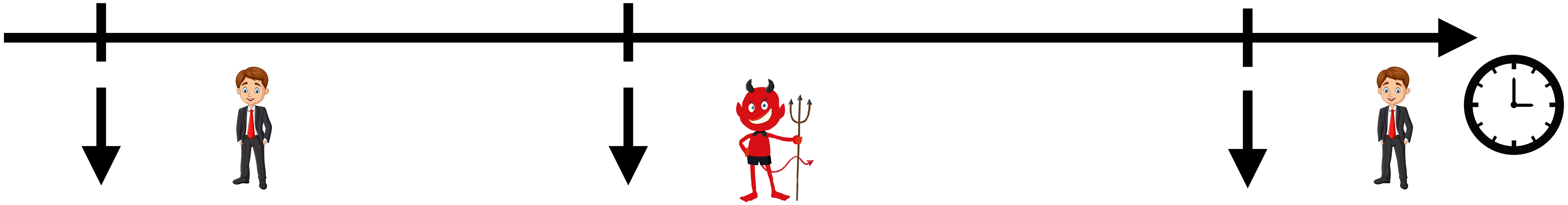
Part 2 **Continuous Recourse Decisions**

Two-Stage Models

- **Decision Rules**
- Lower Bounds
- Benders' Decomposition
- Column-and-Constraint Generation
- Iterative Partitioning
- Fourier-Motzkin Elimination

Delage and Iancu. *Robust Multistage Decision Making*. INFORMS Tutorials in OR, 2015.
Yanıkoğlu, Gorissen and den Hertog. *Adjustable Robust Optimization–A Survey and Tutorial*. EJOR, 2019.
Georghiou, Tsoukalas and W. *On the Optimality of Affine Decision Rules in Distributionally Robust Optimization*. OR, 2025.

Consider again the two-stage robust optimization problem:



$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \;+\; \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \;+\; \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

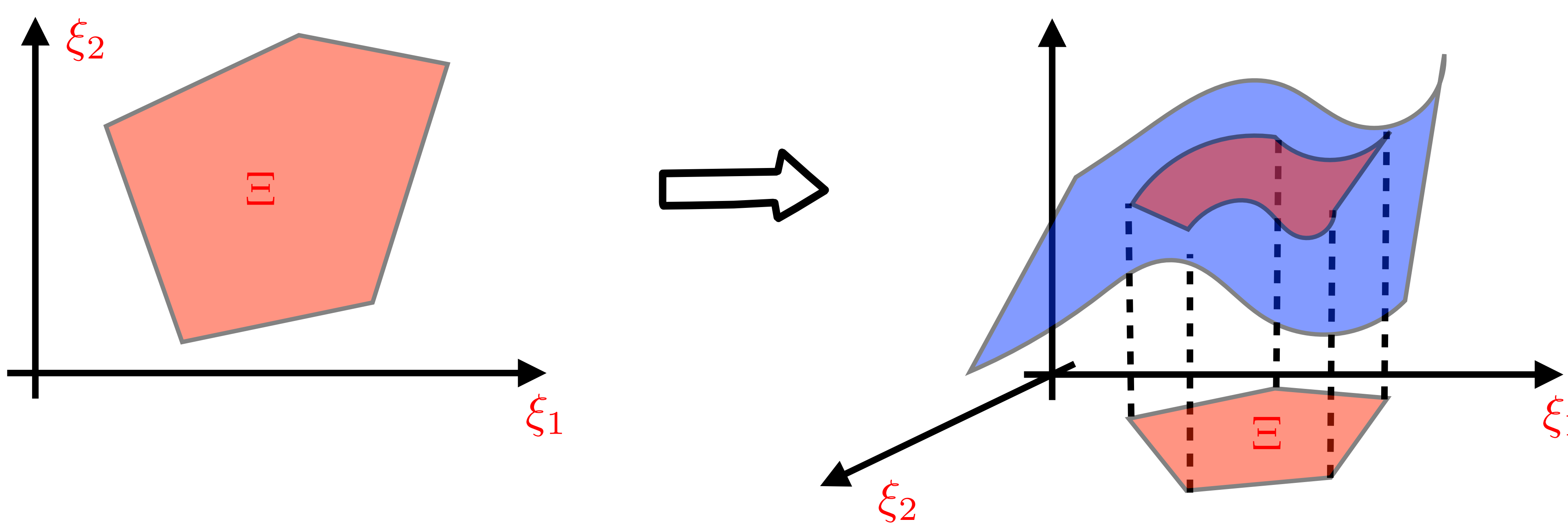
$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Decision Rules

Move second-stage decisions to first stage via decision rules:

$$\min \; \boldsymbol{c}^\top \boldsymbol{x} \quad \text{s.t.} \; \boldsymbol{x} \in \mathcal{X} \quad + \quad \max \; \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \quad \text{s.t.} \; \boldsymbol{\xi} \in \Xi \quad + \quad \min \; \boldsymbol{r}^\top \boldsymbol{y} \quad \text{s.t.} \; \boldsymbol{y} \in \mathcal{Y}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$



The two-stage problem then turns into a single-stage problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \\ & \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi})$$

The two-stage problem then turns into a single-stage problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \\ & \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi})$$

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \; + \; \max\limits_{\boldsymbol{\xi} \in \Xi} \; \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

The two-stage problem then turns into a single-stage problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \\ & \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi})$$

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max\limits_{\boldsymbol{\xi} \in \Xi} \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boxed{\boldsymbol{y} : \Xi \mapsto \mathcal{Y}} & \end{array}$$

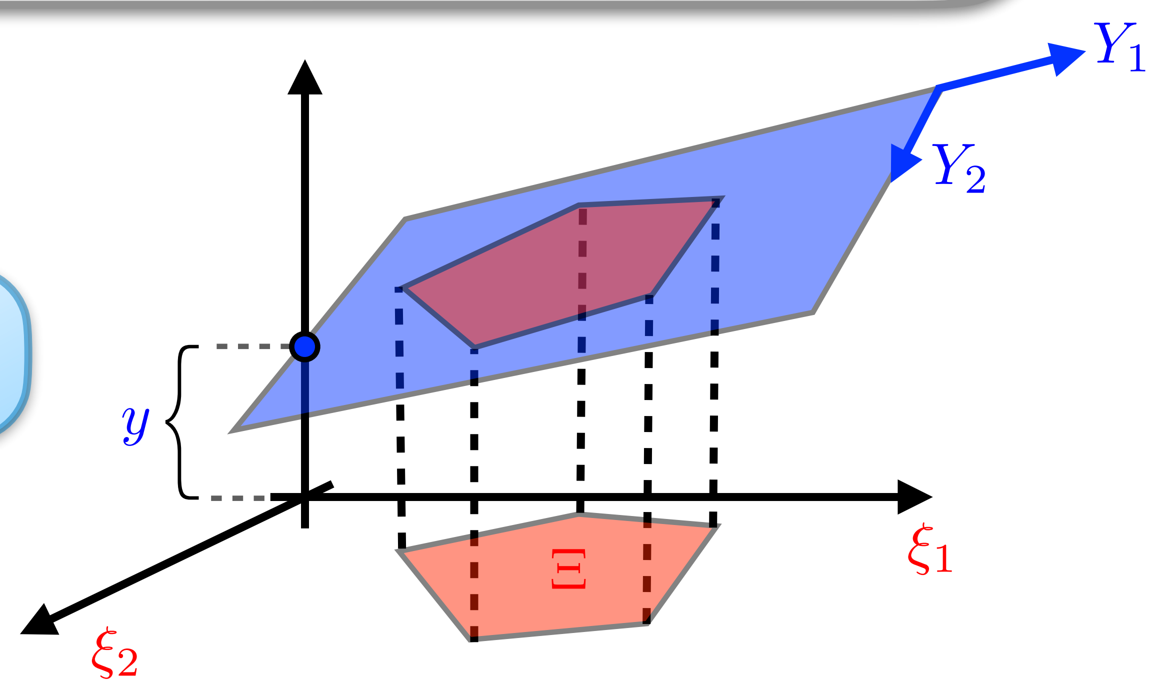
👍 single stage

👎 functional decisions

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

$$\Downarrow$$

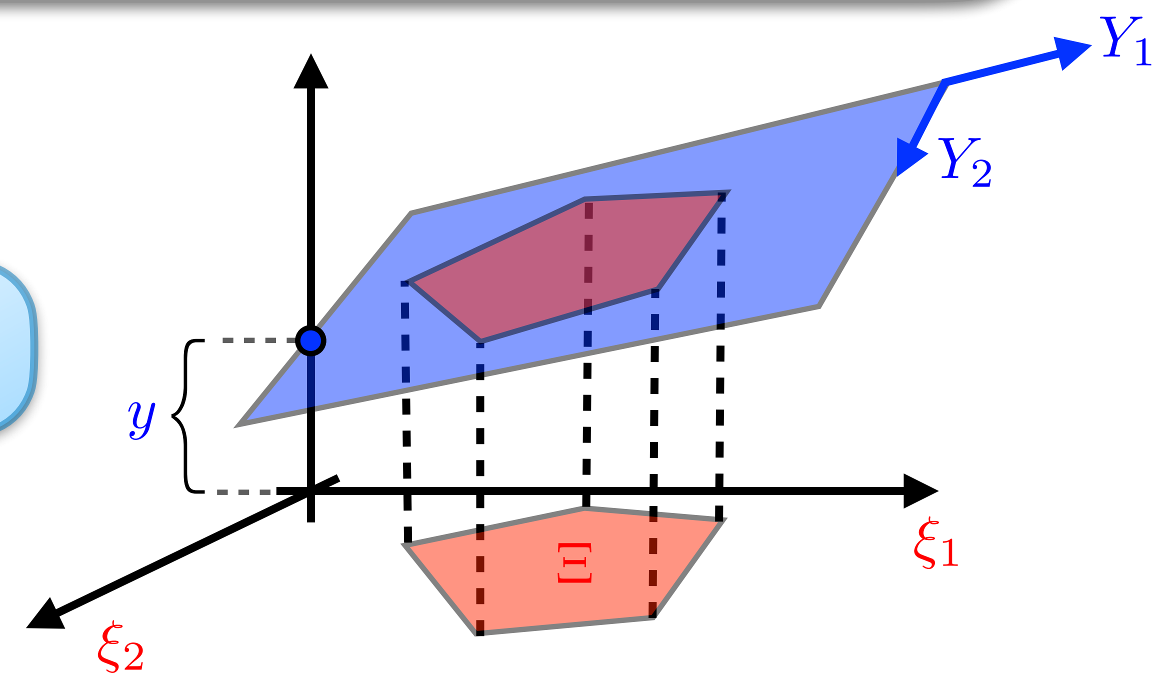
$$\boldsymbol{y}(\boldsymbol{\xi}) \stackrel{!}{=} \boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}$$



Decision Rules: Affine Decision Rules

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} &
\end{array}
$$

$$
\boldsymbol{y}(\boldsymbol{\xi}) \overset{!}{=} \boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}
$$



$$
\begin{array}{lll}
\underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}[\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& [\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] \in \mathcal{Y} & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) &
\end{array}
$$

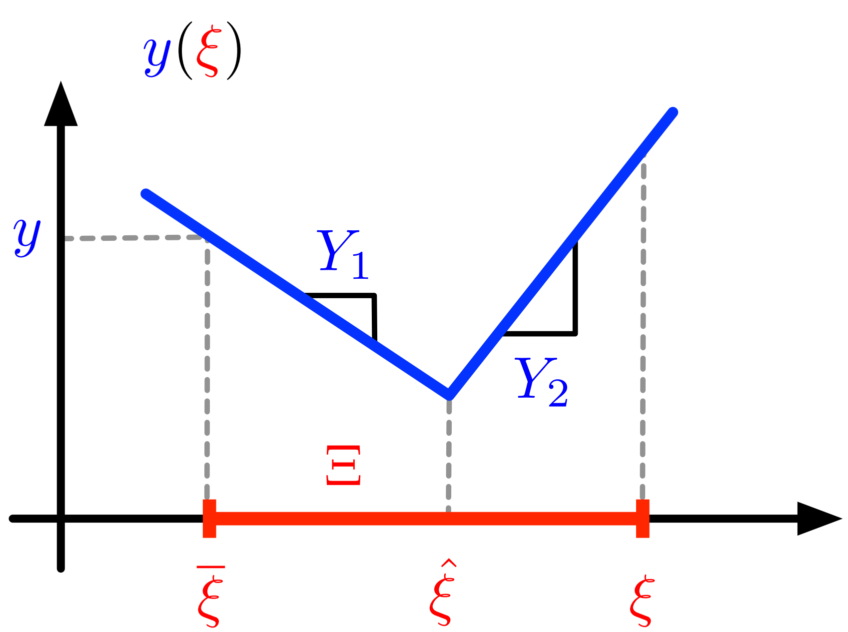
single-stage RO problem

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

Decision Rules: Nonlinear Decision Rules

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} &
\end{array}
$$

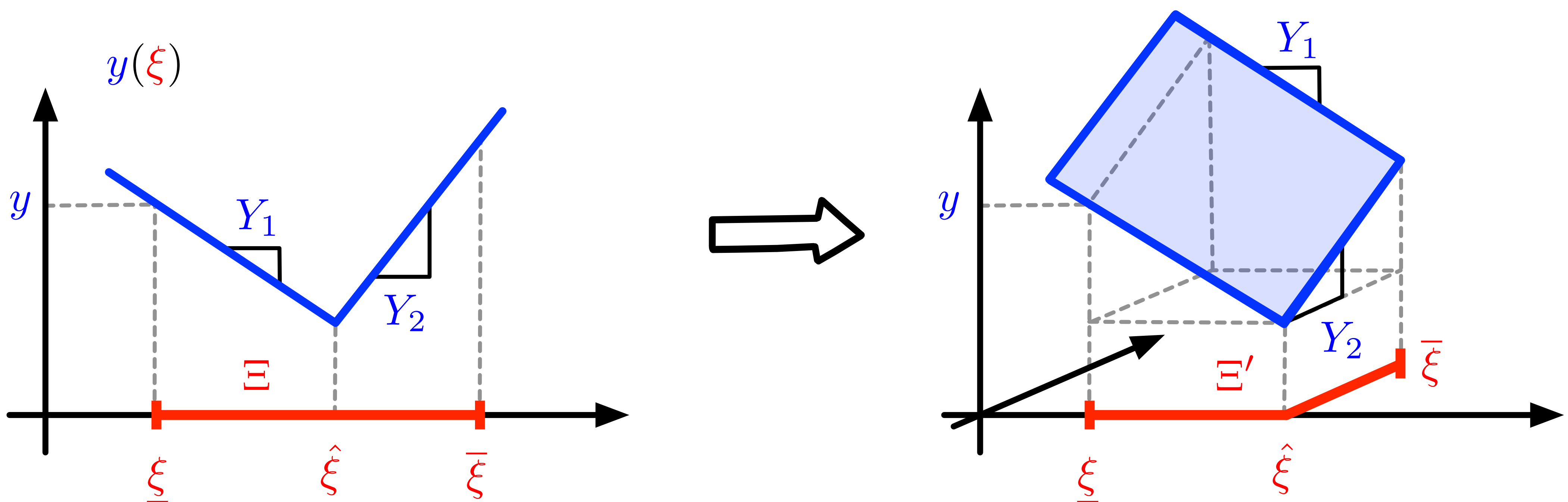
$$
y(\xi) = Y_1 \left[\min\{\xi, \hat{\xi}\} - \underline{\xi}\right] + Y_2 \left[\max\{\xi, \hat{\xi}\} - \hat{\xi}\right] + y
$$



Decision Rules: Nonlinear Decision Rules

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} &
\end{array}
$$

$$
y(\xi) = Y_1 \left[\min\{\xi, \hat{\xi}\} - \underline{\xi}\right] + Y_2 \left[\max\{\xi, \hat{\xi}\} - \hat{\xi}\right] + y
$$



$$
y(\boldsymbol{\xi}') = Y_1 \xi'_1 + Y_2 \xi'_2 + y \quad \text{with} \quad \boldsymbol{\xi}' \in \Xi' = \left\{ \boldsymbol{\xi}' = \begin{bmatrix} \min\{\xi, \hat{\xi}\} - \underline{\xi} \\ \max\{\xi, \hat{\xi}\} - \hat{\xi} \end{bmatrix} \; : \; \xi \in [\underline{\xi}, \overline{\xi}] \right\}
$$

Decision Rules: Nonlinear Decision Rules

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

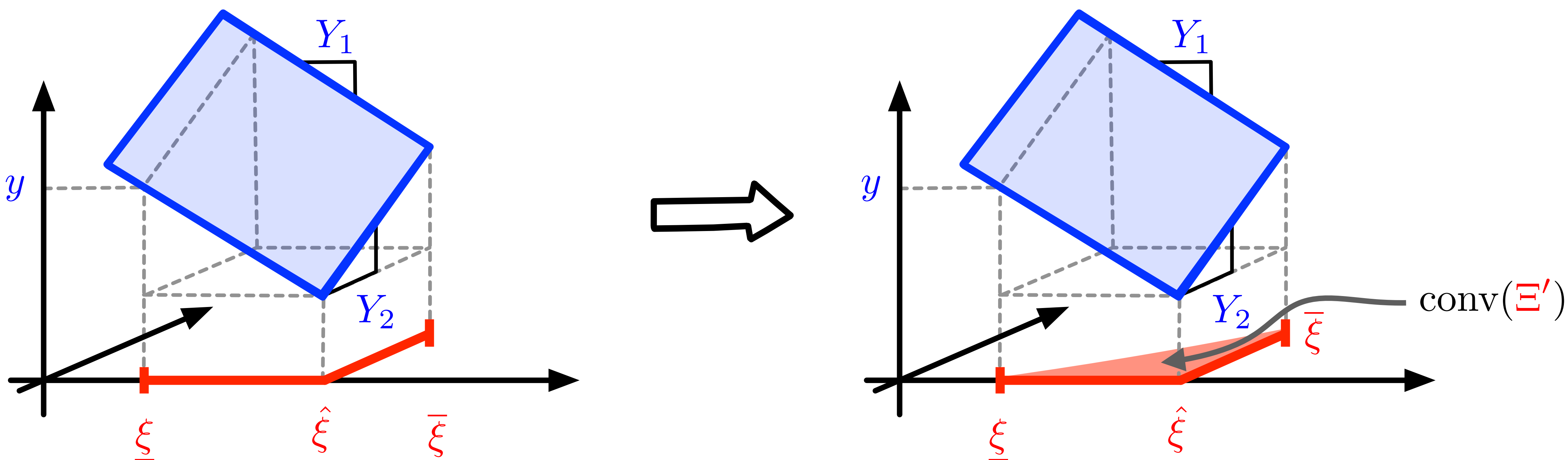
nonlinear decision rules

non-convex uncertainty set

$$\begin{array}{lll} \underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi}')^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y} \boldsymbol{\xi}' + \boldsymbol{y}] & \forall \boldsymbol{\xi}' \in \Xi' \\ & \boldsymbol{T}(\boldsymbol{\xi}')\, \boldsymbol{x} + \boldsymbol{W} [\boldsymbol{Y} \boldsymbol{\xi}' + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}') & \forall \boldsymbol{\xi}' \in \Xi' \\ & [\boldsymbol{Y} \boldsymbol{\xi}' + \boldsymbol{y}] \in \mathcal{Y} & \forall \boldsymbol{\xi}' \in \Xi' \\ & \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) & \end{array}$$

Decision Rules: Nonlinear Decision Rules

$$
\begin{array}{lll}
\underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi}')^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] & \forall \boldsymbol{\xi}' \in \Xi' \\
& \boldsymbol{T}(\boldsymbol{\xi}')\,\boldsymbol{x} + \boldsymbol{W}\,[\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}') & \forall \boldsymbol{\xi}' \in \Xi' \\
& [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \in \mathcal{Y} & \forall \boldsymbol{\xi}' \in \Xi' \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) &
\end{array}
$$



$y(\boldsymbol{\xi}') = Y_1 \xi_1' + Y_2 \xi_2' + y$ with $\boldsymbol{\xi}' \in \Xi' = \left\{ \boldsymbol{\xi}' = \begin{bmatrix} \min\{\xi, \hat{\xi}\} - \underline{\xi} \\ \max\{\xi, \hat{\xi}\} - \hat{\xi} \end{bmatrix} \; : \; \xi \in [\underline{\xi}, \overline{\xi}] \right\}$

Decision Rules: Nonlinear Decision Rules

nonlinear decision rules

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

non-convex uncertainty set

$$\begin{array}{lll} \underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi}')^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] & \forall \boldsymbol{\xi}' \in \Xi' \\ & \boldsymbol{T}(\boldsymbol{\xi}')\,\boldsymbol{x} + \boldsymbol{W}[\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}') & \forall \boldsymbol{\xi}' \in \Xi' \\ & [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \in \mathcal{Y} & \forall \boldsymbol{\xi}' \in \Xi' \\ & \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) & \end{array}$$

complicated representation

$$\begin{array}{lll} \underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi}')^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] & \forall \boldsymbol{\xi}' \in \operatorname{conv}(\Xi') \\ & \boldsymbol{T}(\boldsymbol{\xi}')\,\boldsymbol{x} + \boldsymbol{W}[\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}') & \forall \boldsymbol{\xi}' \in \operatorname{conv}(\Xi') \\ & [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \in \mathcal{Y} & \forall \boldsymbol{\xi}' \in \operatorname{conv}(\Xi') \\ & \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) & \end{array}$$

Decision Rules: Nonlinear Decision Rules

nonlinear decision rules

$$\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} &
\end{array}$$

non-convex uncertainty set

$$\begin{array}{lll}
\underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi}')^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] & \forall \boldsymbol{\xi}' \in \Xi' \\
& \boldsymbol{T}(\boldsymbol{\xi}')\,\boldsymbol{x} + \boldsymbol{W}[\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}') & \forall \boldsymbol{\xi}' \in \Xi' \\
& & \forall \boldsymbol{\xi}' \in \Xi'
\end{array}$$

use exact reformulations
or outer approximations
from global optimization

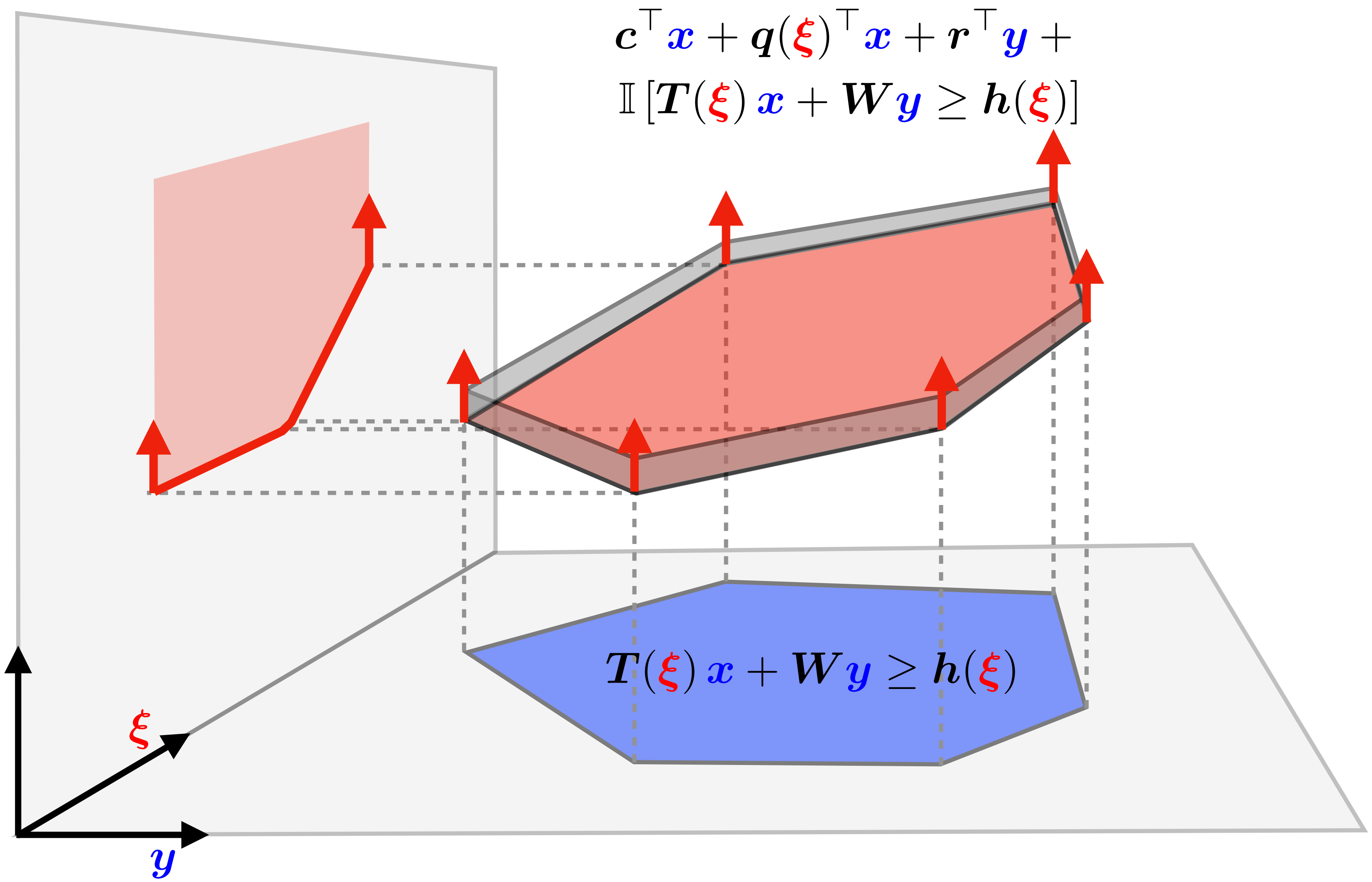
complicated representation

$$\begin{array}{lll}
\underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi}')^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] & \forall \boldsymbol{\xi}' \in \operatorname{conv}(\Xi') \\
& \boldsymbol{T}(\boldsymbol{\xi}')\,\boldsymbol{x} + \boldsymbol{W}[\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}') & \forall \boldsymbol{\xi}' \in \operatorname{conv}(\Xi') \\
& [\boldsymbol{Y}\boldsymbol{\xi}' + \boldsymbol{y}] \in \mathcal{Y} & \forall \boldsymbol{\xi}' \in \operatorname{conv}(\Xi') \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) &
\end{array}$$

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

Recall that a worst-case $\boldsymbol{\xi}^\star \in \Xi$ will be an extreme point $\boldsymbol{\xi}^\star \in \operatorname{ext} \Xi$:



Decision Rules: Iterative Liftings

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} &
\end{array}
$$

1

affine decision rules

$$
\begin{array}{lll}
\underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}[\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& [\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] \in \mathcal{Y} & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) &
\end{array}
$$

 tractable

 suboptimal

Decision Rules: Iterative Liftings

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} &
\end{array}
$$

1

affine decision rules

$$
\begin{array}{lll}
\underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top [\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}[\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& [\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] \in \mathcal{Y} & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) &
\end{array}
$$

tractable

suboptimal

2

extreme point formulation

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \operatorname{ext}(\Xi) \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \operatorname{ext}(\Xi) \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \operatorname{ext}(\Xi) \mapsto \mathcal{Y} &
\end{array}
$$

optimal

intractable

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

Split up Ξ into affine DR part Ξ_A and extreme point part Ξ_S:

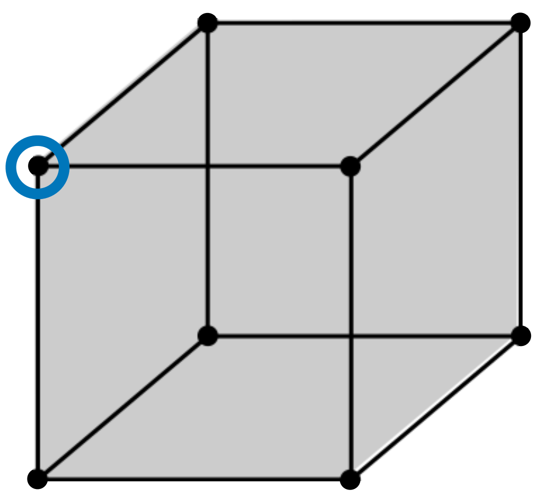
$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_A(\boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi} \in \Xi_A \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}_A(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi_A \\ & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_S(\boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi} \in \Xi_S \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}_S(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi_S \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}_S : \Xi_S \mapsto \mathcal{Y}, \quad \boldsymbol{y}_A : \Xi_A \mapsto \mathcal{Y} \end{array}$$

affine decision rules + **extreme point formulation**

Decision Rules: Iterative Liftings

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_A(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_A \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}_A(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_A \\ & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_S(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_S \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}_S(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_S \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}_S : \Xi_S \mapsto \mathcal{Y}, \quad \boldsymbol{y}_A : \Xi_A \mapsto \mathcal{Y} & \end{array}$$

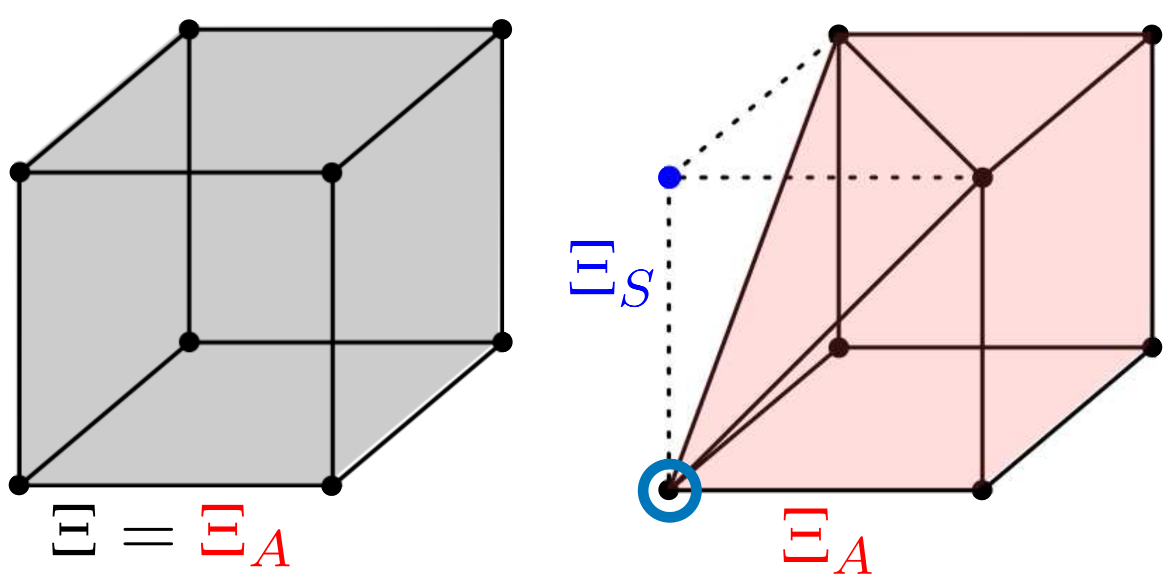
affine decision rules + **extreme point formulation**



$\Xi = \Xi_A$

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_A(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_A \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}_A(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_A \\ & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_S(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_S \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}_S(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_S \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}_S : \Xi_S \mapsto \mathcal{Y}, \quad \boldsymbol{y}_A : \Xi_A \mapsto \mathcal{Y} & \end{array}$$

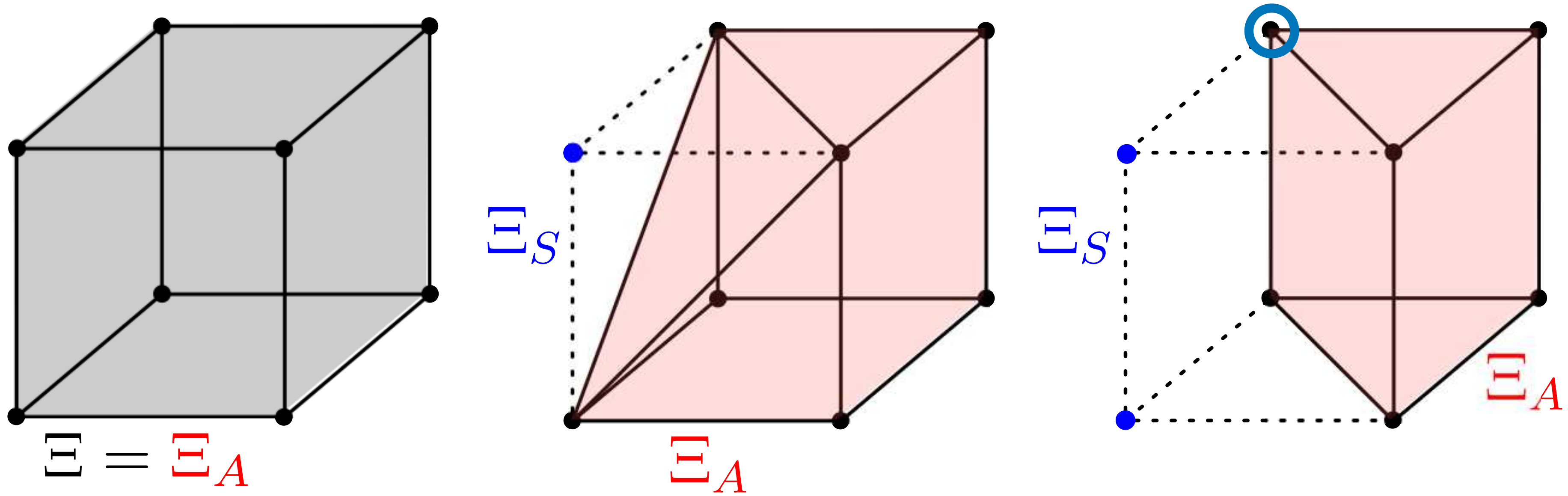
affine decision rules + **extreme point formulation**



$\Xi_A = \text{conv}([\text{ext } \Xi \setminus \Xi_S)$ such that $\text{conv}(\Xi_S \cup \Xi_A) = \Xi$

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_A(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_A \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}_A(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_A \\ & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_S(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_S \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}_S(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_S \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}_S : \Xi_S \mapsto \mathcal{Y}, \quad \boldsymbol{y}_A : \Xi_A \mapsto \mathcal{Y} & \end{array}$$

affine decision rules + **extreme point formulation**

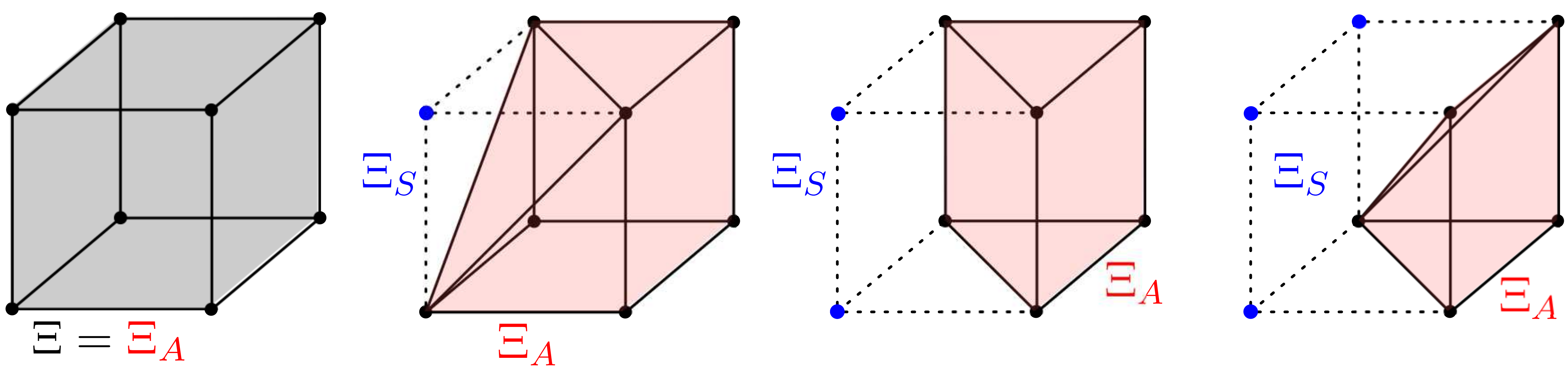


$\Xi_A = \text{conv}([\text{ext}\ \Xi \setminus \Xi_S)$ such that $\text{conv}(\Xi_S \cup \Xi_A) = \Xi$

Decision Rules: Iterative Liftings

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\
\text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_A(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_A \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}_A(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_A \\
& \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}_S(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_S \\
& \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}_S(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi_S \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}_S : \Xi_S \mapsto \mathcal{Y}, \quad \boldsymbol{y}_A : \Xi_A \mapsto \mathcal{Y} &
\end{array}
$$

affine decision rules + **extreme point formulation**



$\Xi_A = \text{conv}([\text{ext}\ \Xi \setminus \Xi_S)$ such that $\text{conv}(\Xi_S \cup \Xi_A) = \Xi$

Part 2 **Continuous Recourse Decisions**

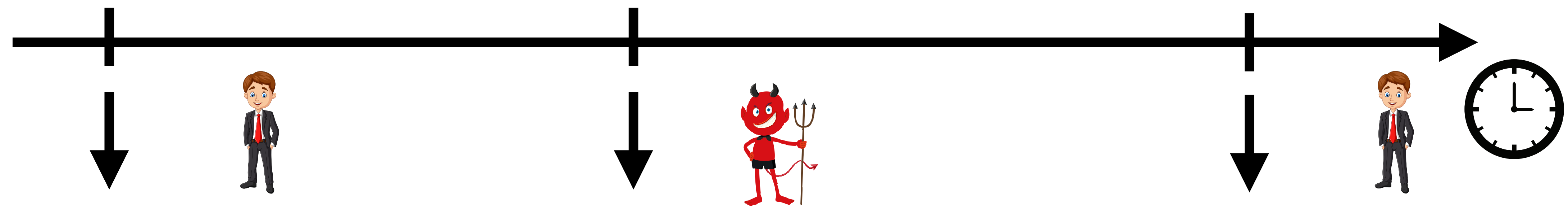
Two-Stage Models

- Decision Rules
- **Lower Bounds**
- Benders' Decomposition
- Column-and-Constraint Generation
- Iterative Partitioning
- Fourier-Motzkin Elimination

Hadjiyiannis, Goulart and Kuhn. *A Scenario Approach for Estimating the Suboptimality of Linear Decision Rules in Two-Stage Robust Optimization*. IEEE CDC 2011.

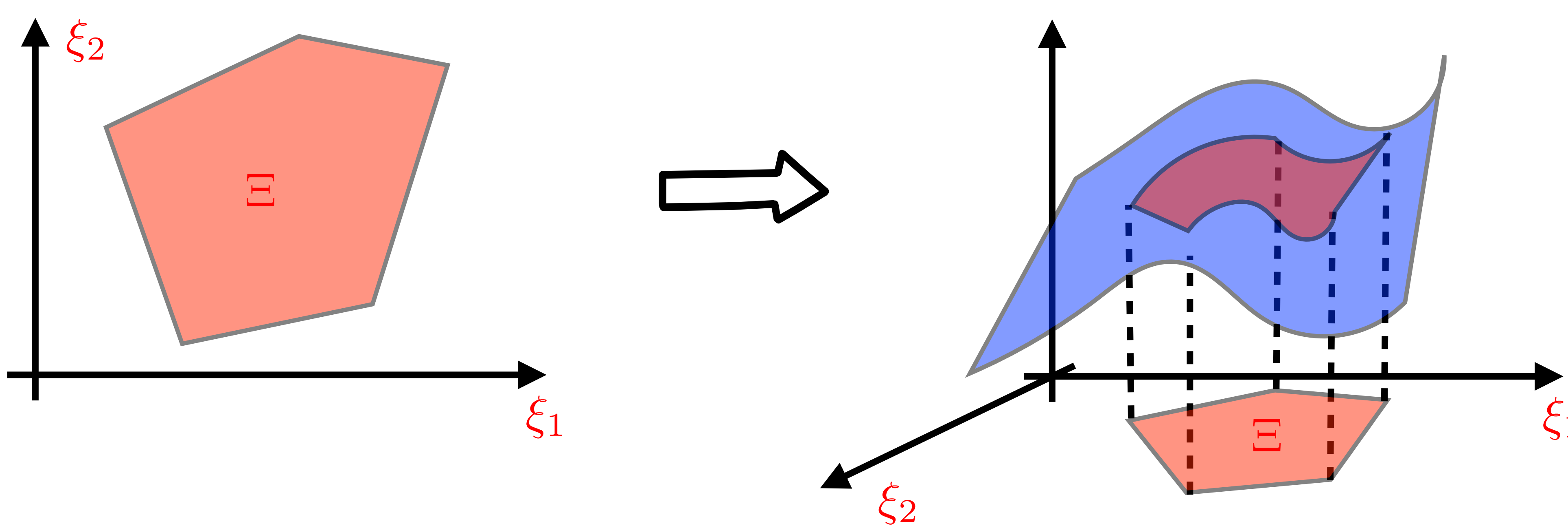
Lower Bounds: The Hadjiyiannis Scenario Relaxation

Recall the two-stage robust optimization problem:

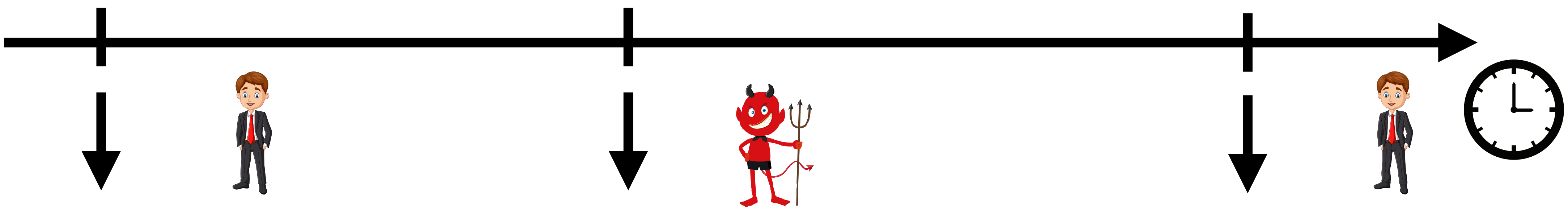


$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$



Recall the two-stage robust optimization problem:



$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \xcancel{\begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array}} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

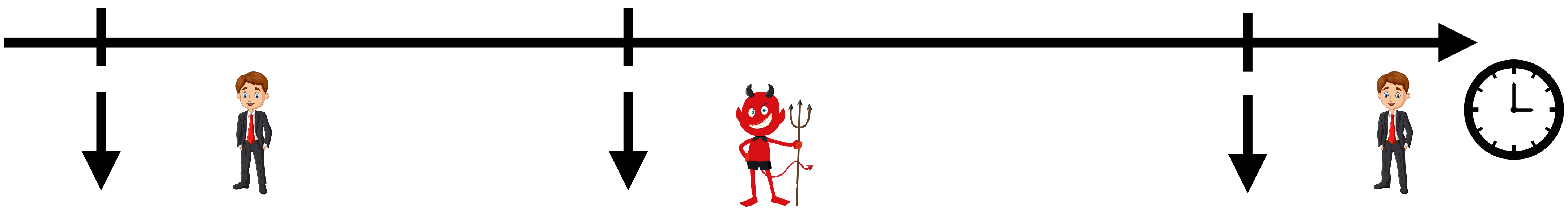
$$\Downarrow$$

$$\begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \hat{\Xi} \end{array}$$

Replace Ξ with a finite subset $\hat{\Xi}$!

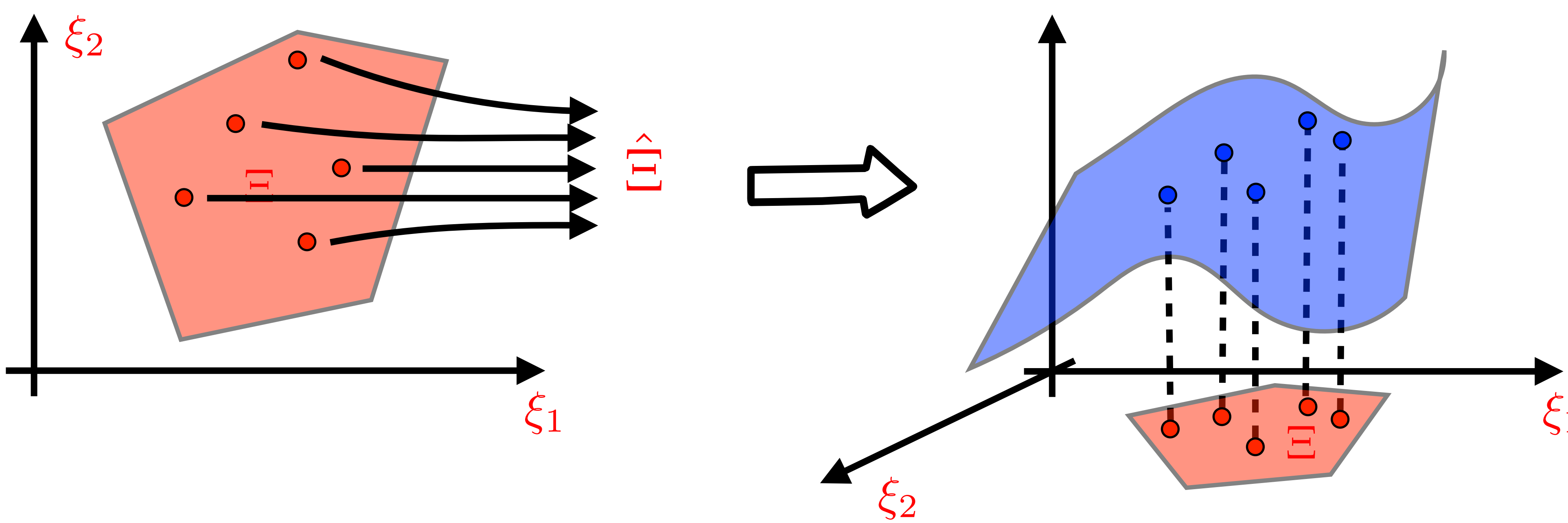
Lower Bounds: The Hadjiyiannis Scenario Relaxation

Recall the two-stage robust optimization problem:



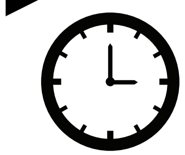
$$\begin{array}{ll}\min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X}\end{array} \quad + \quad \begin{array}{ll}\max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \hat{\Xi}\end{array} \quad + \quad \begin{array}{ll}\min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y}\end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$



Lower Bounds: The Hadjiyiannis Scenario Relaxation

Recall the two-stage robust optimization problem:

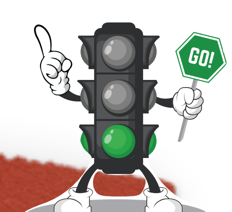


$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \hat{\Xi} \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\hat{\boldsymbol{\xi}})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\hat{\boldsymbol{\xi}}) & \forall \hat{\boldsymbol{\xi}} \in \hat{\Xi} \\ & \boldsymbol{T}(\hat{\boldsymbol{\xi}})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\hat{\boldsymbol{\xi}}) \geq \boldsymbol{h}(\hat{\boldsymbol{\xi}}) & \forall \hat{\boldsymbol{\xi}} \in \hat{\Xi} \\ & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}(\hat{\boldsymbol{\xi}}) \in \mathcal{Y}, \; \hat{\boldsymbol{\xi}} \in \hat{\Xi} & \end{array}$$

finite-dimensional LP



Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

We can dualize this problem and obtain:

$$\begin{array}{ll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} \\ & \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad\qquad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+ \end{array}$$

Recall the decision rule formulation of the two-stage RO problem:

$$
\begin{array}{ll}
\underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}
\end{array}
$$

We can dualize this problem and obtain:

$$
\begin{array}{ll}
\underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\
\text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} \\
& \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+
\end{array}
$$

?

HOW?

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

We can dualize this problem and obtain:

$$\begin{array}{lll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \text{ext}\,\Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) & \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \text{ext}\,\Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} & \\ & \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+ & \end{array}$$

1 primal decision rule formulation

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

We can dualize this problem and obtain:

$$\begin{array}{ll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \text{ext}\,\Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \text{ext}\,\Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} \\ & \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+ \end{array}$$

1 primal decision rule formulation

2 primal extreme point formulation

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

We can dualize this problem and obtain:

$$\begin{array}{ll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} \\ & \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+ \end{array}$$

1 primal decision rule formulation → 2 primal extreme point formulation → 3 dual extreme point formulation

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

We can dualize this problem and obtain:

$$\begin{array}{lll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) & \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} & \\ & \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+ & \end{array}$$

1. primal decision rule formulation
2. primal extreme point formulation
3. dual extreme point formulation
4. dual decision rule formulation

Recall the decision rule formulation of the two-stage RO problem:

$$
\begin{array}{ll}
\underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}
\end{array}
$$

We can dualize this problem and obtain:

$$
\begin{array}{ll}
\underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\
\text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} \\
& \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+
\end{array}
$$

The dual formulation raises two challenges:

1. It has infinitely many constraints and variables.

Lower Bounds: Dual Decision Rules

Recall the decision rule formulation of the two-stage RO problem:

$$
\begin{array}{ll}
\underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}
\end{array}
$$

We can dualize this problem and obtain:

$$
\begin{array}{ll}
\underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\
\text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} \\
& \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+
\end{array}
$$

The dual formulation raises two challenges:

1. It has infinitely many constraints and variables.

 affine decision rules + RO trick

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

We can dualize this problem and obtain:

$$\begin{array}{ll} \underset{\boldsymbol{\lambda}}{\text{maximize}} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} \\ & \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+ \end{array}$$

The dual formulation raises two challenges:

1. It has infinitely many constraints and variables.
2. It contains sums over exponentially many terms.

Lower Bounds: Dual Decision Rules

Recall the decision rule formulation of the two-stage RO problem:

$$
\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}
\end{aligned}
$$

We can dualize this problem and obtain:

$$
\begin{aligned}
\underset{\boldsymbol{\lambda}}{\text{maximize}} \quad & \sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{h}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) \\
\text{subject to} \quad & \sum_{\boldsymbol{\xi} \in \operatorname{ext} \Xi} \boldsymbol{T}(\boldsymbol{\xi})^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{q} \\
& \boldsymbol{W}^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{0} \qquad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{\lambda} : \Xi \mapsto \mathbb{R}^m_+
\end{aligned}
$$

The dual formulation raises two challenges:

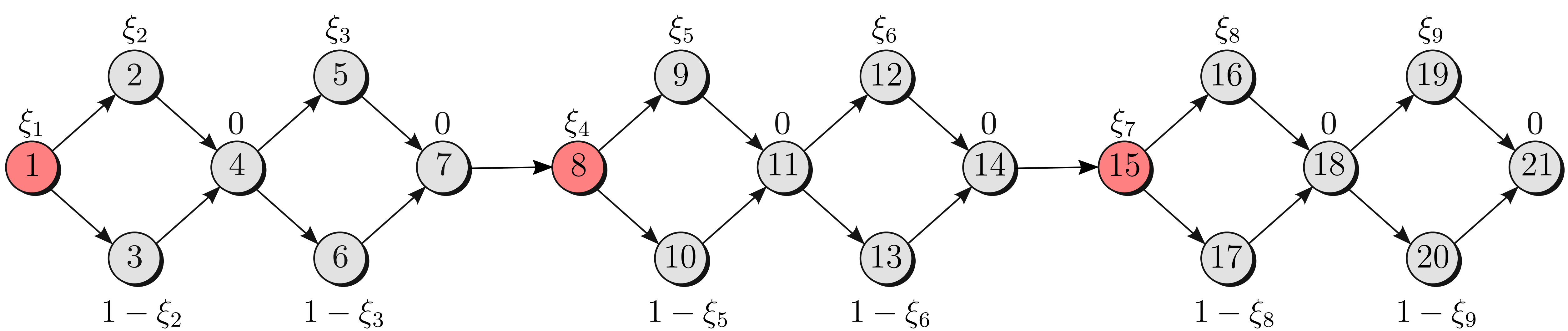
closed-form expressions for structured $\boldsymbol{\lambda}$ and Ξ

2. It contains sums over exponentially many terms.

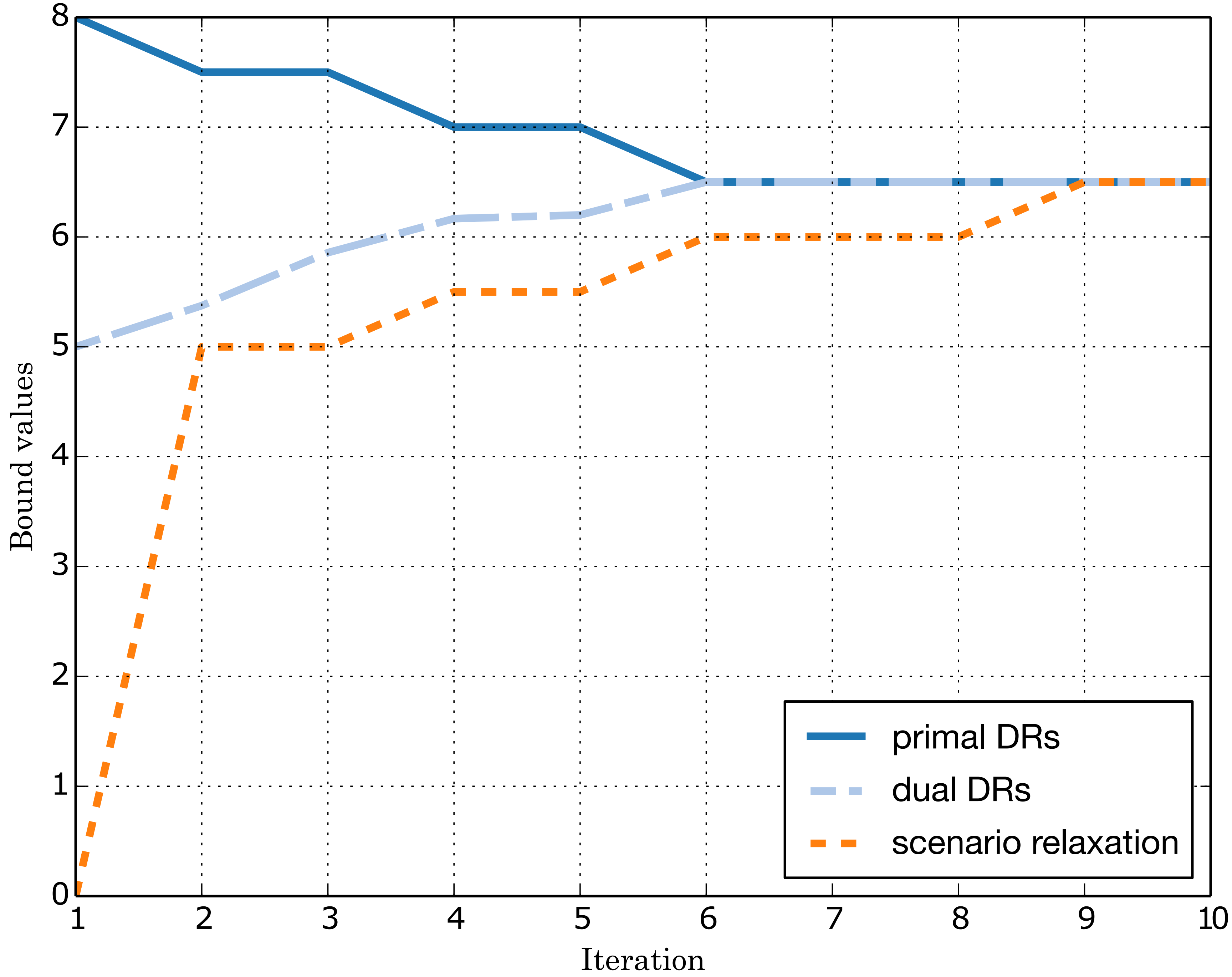
Numerical example: Estimating the makespan of a project

$$
\begin{array}{ll}
\underset{\boldsymbol{x},\boldsymbol{y},\tau}{\text{minimize}} & \tau \\
\text{subject to} & \\
& \left.\begin{array}{ll}
\tau \geq y_n(\boldsymbol{\xi}) + d_n(\boldsymbol{\xi}) & \\
y_j(\boldsymbol{\xi}) \geq y_i(\boldsymbol{\xi}) + d_i(\boldsymbol{\xi}) & \forall (i,j) \in E \\
y_i(\boldsymbol{\xi}) = x_i & \forall i \in V_0
\end{array}\right\} \quad \forall \boldsymbol{\xi} \in \Xi \\
& \tau \in \mathbb{R}_+, \quad x_i \in \mathbb{R}_+,\ i \in V_0, \quad \boldsymbol{y} : \Xi \mapsto \mathbb{R}^n_+.
\end{array}
$$

Example project:



Comparison: Scenario relaxation vs dual affine decision rules



Part 2 **Continuous Recourse Decisions**

Two-Stage Models

- Decision Rules
- Lower Bounds
- **Benders' Decomposition**
- Column-and-Constraint Generation
- Iterative Partitioning
- Fourier-Motzkin Elimination

Thiele, Terry and Epelman. *Robust Linear Optimization With Recourse*. Technical Report, 2010.

Benders' Decomposition

Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

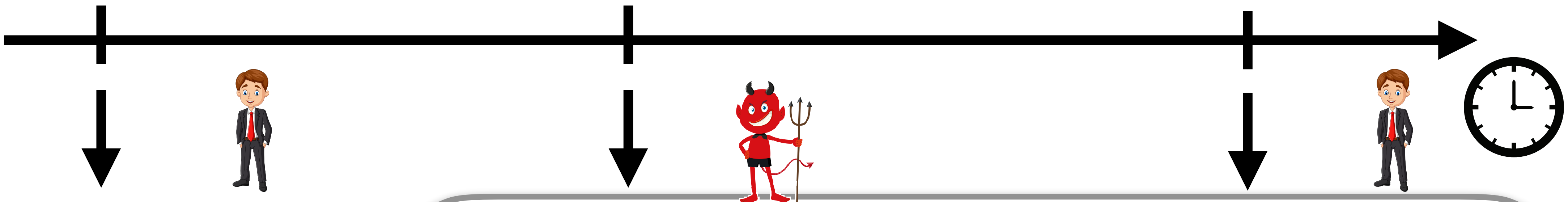
$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top} \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Recall the two-stage robust optimization problem:



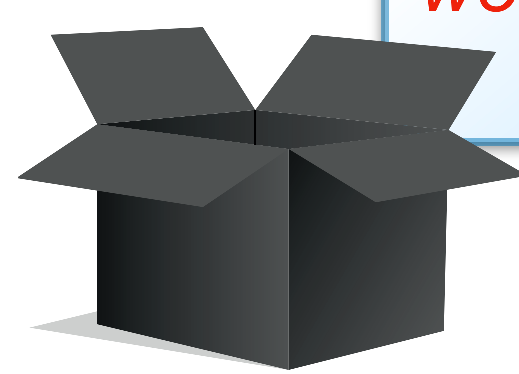
$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & x \in \mathcal{X} \end{array} \; + \; \underbrace{\begin{array}{ll} \max & \cancel{q(\xi)^\top x} \\ \text{s.t.} & \xi \in \Xi \end{array} \; + \; \begin{array}{ll} \min & r^\top y \\ \text{s.t.} & y \in \mathcal{Y} \end{array}}_{\mathcal{Q}(x)}$$

$$T(\xi)\, x + W y \geq h(\xi)$$

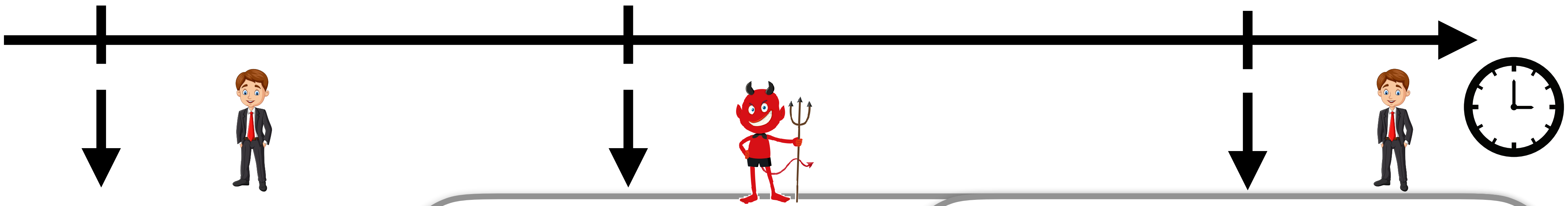
We can represent it as a single-stage optimization problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^\top x + \boxed{\mathcal{Q}(x)} \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

"Black-box" function
that computes the
worst-case second-stage
costs of a fixed x!



Recall the two-stage robust optimization problem:



$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \; + \; \underbrace{\begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \; + \; \underbrace{\begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi}) \end{array}}_{Q(\boldsymbol{x};\boldsymbol{\xi})}}_{\mathcal{Q}(\boldsymbol{x})}$$

We can represent it as a single-stage optimization problem:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$

with

$$\mathcal{Q}(\boldsymbol{x}) = \max_{\boldsymbol{\xi} \in \Xi} \; Q(\boldsymbol{x};\boldsymbol{\xi})$$

$$Q(\boldsymbol{x};\boldsymbol{\xi}) = \left[\begin{array}{ll} \underset{\boldsymbol{y}}{\text{minimize}} & \boldsymbol{q}^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi}) \\ & \boldsymbol{y} \in \mathcal{Y} \end{array} \right]$$

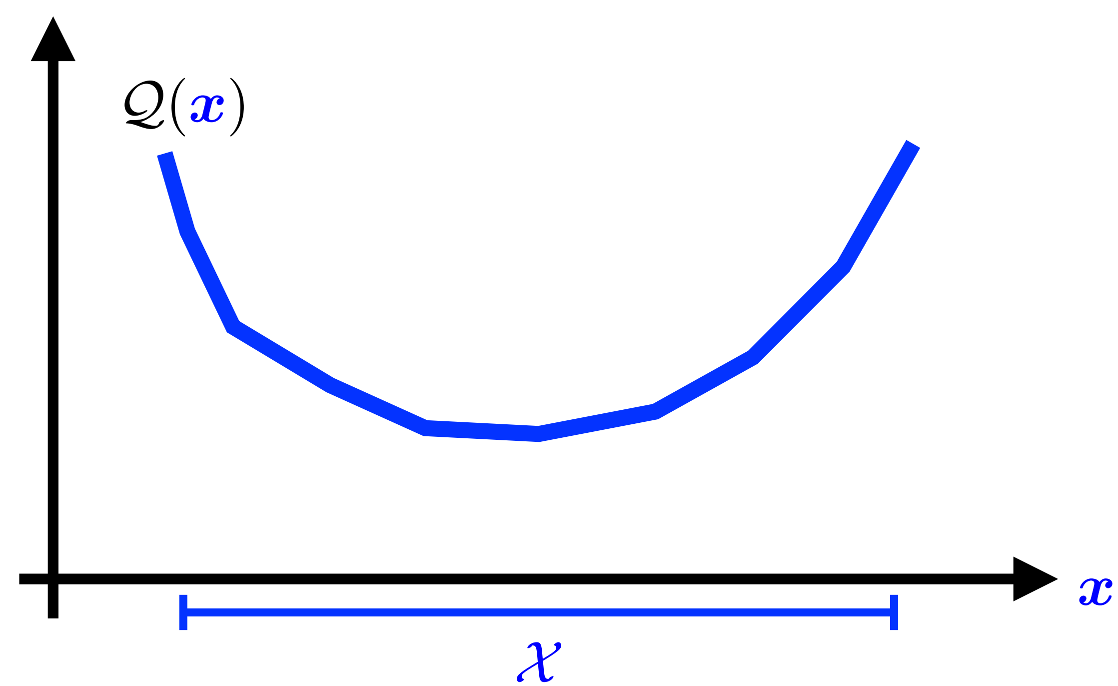
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top} \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Standard arguments show that $\mathcal{Q}(\boldsymbol{x})$ is piecewise affine and convex:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



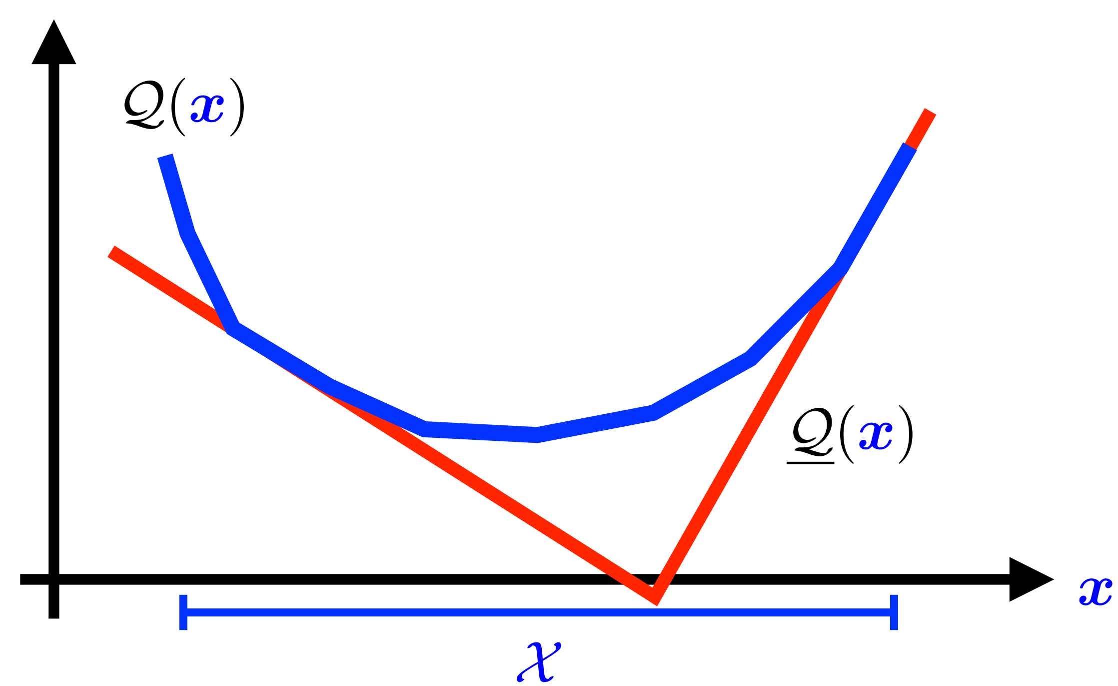
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Benders' decomposition: optimize over & refine lower bounds $\underline{\mathcal{Q}}(\boldsymbol{x})$:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



Benders’ Decomposition

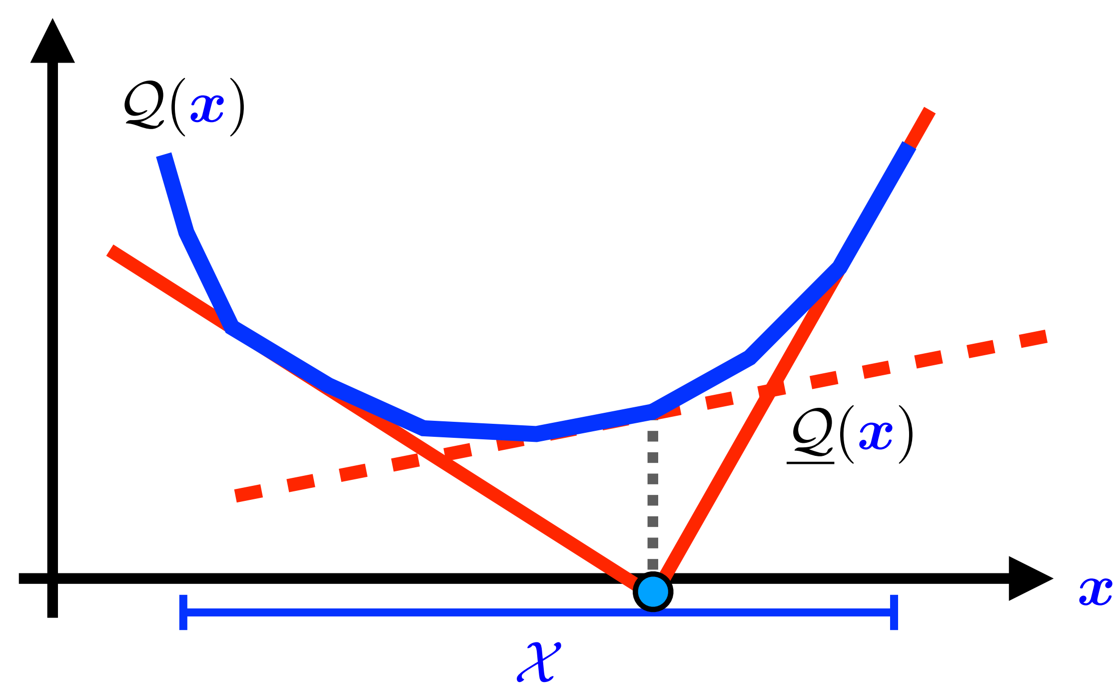
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Benders’ decomposition: optimize over & refine lower bounds $\underline{\mathcal{Q}}(\boldsymbol{x})$:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \underline{\mathcal{Q}}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



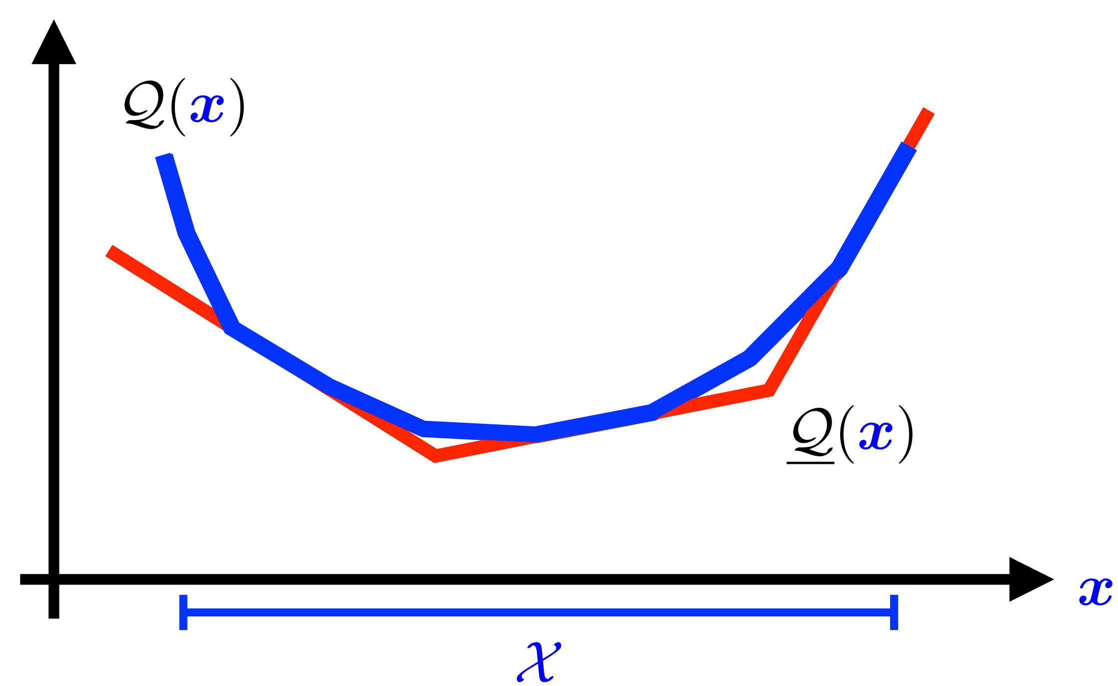
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top} \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Benders' decomposition: optimize over & refine lower bounds $\underline{\mathcal{Q}}(\boldsymbol{x})$:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \underline{\mathcal{Q}}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



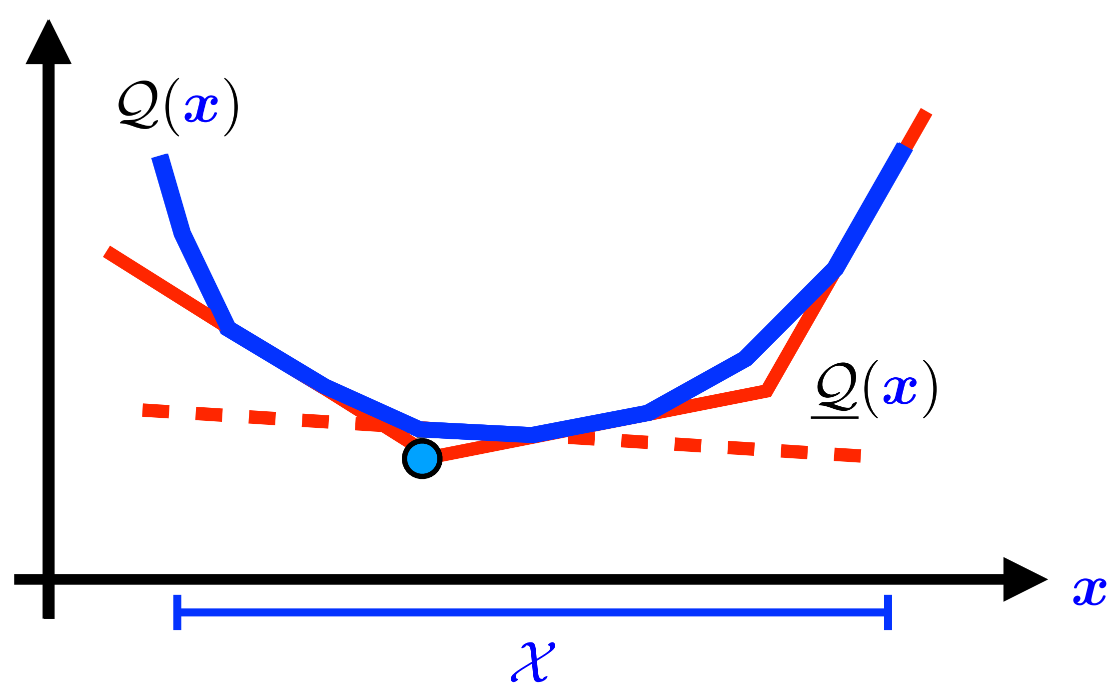
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & x \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{q(\xi)^\top x} \\ \text{s.t.} & \xi \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & r^\top y \\ \text{s.t.} & y \in \mathcal{Y} \end{array}$$

$$T(\xi)\, x + W y \geq h(\xi)$$

Benders' decomposition: optimize over & refine lower bounds $\underline{\mathcal{Q}}(x)$:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^\top x + \mathcal{Q}(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$



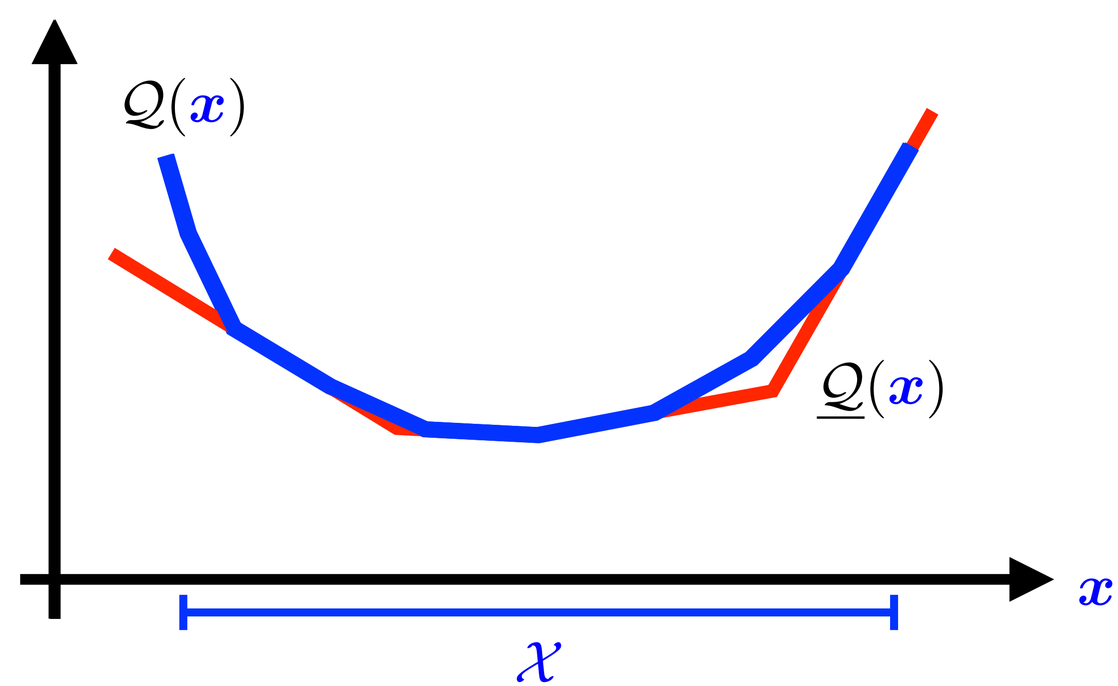
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & x \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{q(\xi)^\top x} \\ \text{s.t.} & \xi \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & r^\top y \\ \text{s.t.} & y \in \mathcal{Y} \end{array}$$

$$T(\xi)\, x + W y \geq h(\xi)$$

Benders' decomposition: optimize over & refine lower bounds $\underline{\mathcal{Q}}(x)$:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^\top x + \underline{\mathcal{Q}}(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$



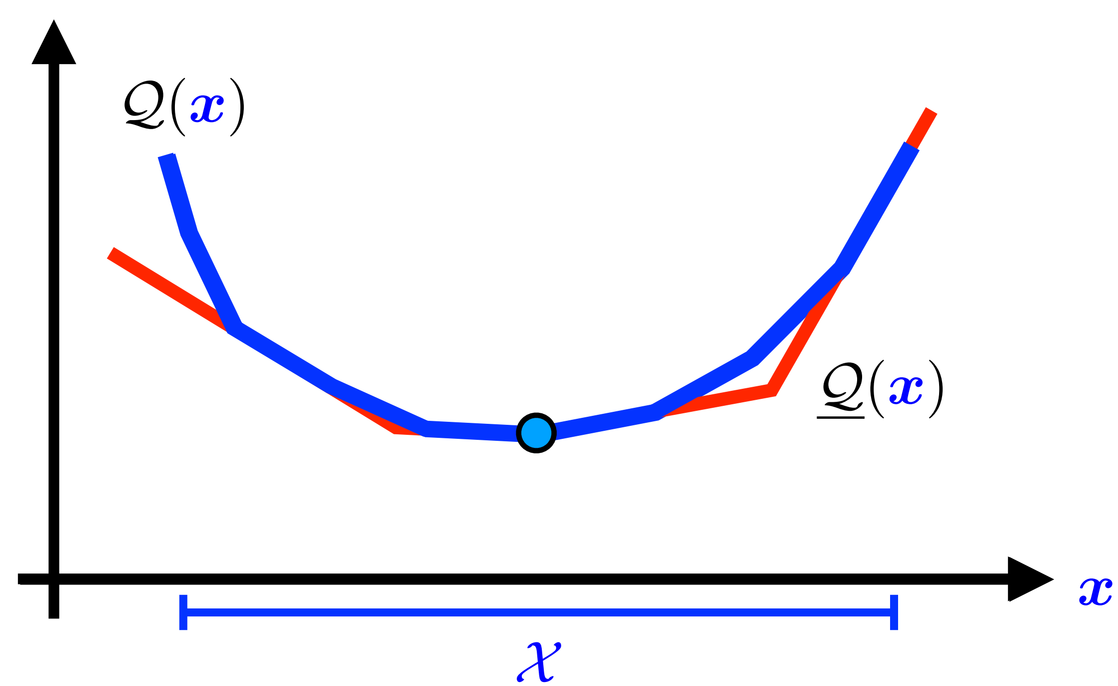
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & x \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{q(\xi)^\top x} \\ \text{s.t.} & \xi \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & r^\top y \\ \text{s.t.} & y \in \mathcal{Y} \end{array}$$

$$T(\xi)\,x + W y \geq h(\xi)$$

Benders' decomposition: optimize over & refine lower bounds $\underline{\mathcal{Q}}(x)$:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^\top x + \underline{\mathcal{Q}}(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$



How can we construct the lower bounds?

$$\left.\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}\right\} \quad \mathcal{Q}(\boldsymbol{x}^\star)$$

Benders' Decomposition

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \, : \, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\iff \quad \begin{array}{ll} \underset{\boldsymbol{\xi},\boldsymbol{\lambda}}{\text{maximize}} & [\boldsymbol{h}(\boldsymbol{\xi}) - \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star]^\top \boldsymbol{\lambda} \\ \text{subject to} & \boldsymbol{W}^\top \boldsymbol{\lambda} \leq \boldsymbol{r}, \quad \boldsymbol{\xi} \in \Xi \end{array}$$

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\Longleftrightarrow \quad \underbrace{\begin{array}{ll} \underset{\boldsymbol{\xi},\boldsymbol{\lambda}}{\text{maximize}} & [\boldsymbol{h}(\boldsymbol{\xi}) - \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star]^\top \boldsymbol{\lambda} \\ \text{subject to} & \boldsymbol{W}^\top \boldsymbol{\lambda} \leq \boldsymbol{r}, \quad \boldsymbol{\xi} \in \Xi \end{array}}$$

optimal value supports $\mathcal{Q}(\boldsymbol{x})$ **at** $\boldsymbol{x} = \boldsymbol{x}^\star$**:** thanks to strong duality

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \underset{\boldsymbol{\xi},\boldsymbol{\lambda}}{\text{maximize}} & [\boldsymbol{h}(\boldsymbol{\xi}) - \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star]^\top \boldsymbol{\lambda} \\ \text{subject to} & \boldsymbol{W}^\top \boldsymbol{\lambda} \leq \boldsymbol{r}, \quad \boldsymbol{\xi} \in \Xi \end{array}$$

- **optimal value supports** $\mathcal{Q}(\boldsymbol{x})$ **at** $\boldsymbol{x} = \boldsymbol{x}^\star$**:** thanks to strong duality

- $[\boldsymbol{h}(\boldsymbol{\xi}^\star) - \boldsymbol{T}(\boldsymbol{\xi}^\star)\,\boldsymbol{x}]^\top \boldsymbol{\lambda}^\star$ **bounds** $\mathcal{Q}(\boldsymbol{x})$ **from below:** since $(\boldsymbol{\xi}^\star, \boldsymbol{\lambda}^\star)$ is feasible but suboptimal at other values of $\boldsymbol{x}$

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\iff \underbrace{\begin{array}{ll} \underset{\boldsymbol{\xi},\boldsymbol{\lambda}}{\text{maximize}} & [\boldsymbol{h}(\boldsymbol{\xi}) - \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star]^\top \boldsymbol{\lambda} \\ \text{subject to} & \boldsymbol{W}^\top \boldsymbol{\lambda} \leq \boldsymbol{r}, \quad \boldsymbol{\xi} \in \Xi \end{array}}$$

- **optimal value supports** $\mathcal{Q}(\boldsymbol{x})$ **at** $\boldsymbol{x} = \boldsymbol{x}^\star$**:** thanks to strong duality

- $[\boldsymbol{h}(\boldsymbol{\xi}^\star) - \boldsymbol{T}(\boldsymbol{\xi}^\star)\,\boldsymbol{x}]^\top \boldsymbol{\lambda}^\star$ **bounds** $\mathcal{Q}(\boldsymbol{x})$ **from below:** since $(\boldsymbol{\xi}^\star, \boldsymbol{\lambda}^\star)$ is feasible but suboptimal at other values of $\boldsymbol{x}$

- **bilinear problem:** solution via global optimization solvers, MILP reformulations (via KKT conditions or for structured uncertainty sets Ξ)

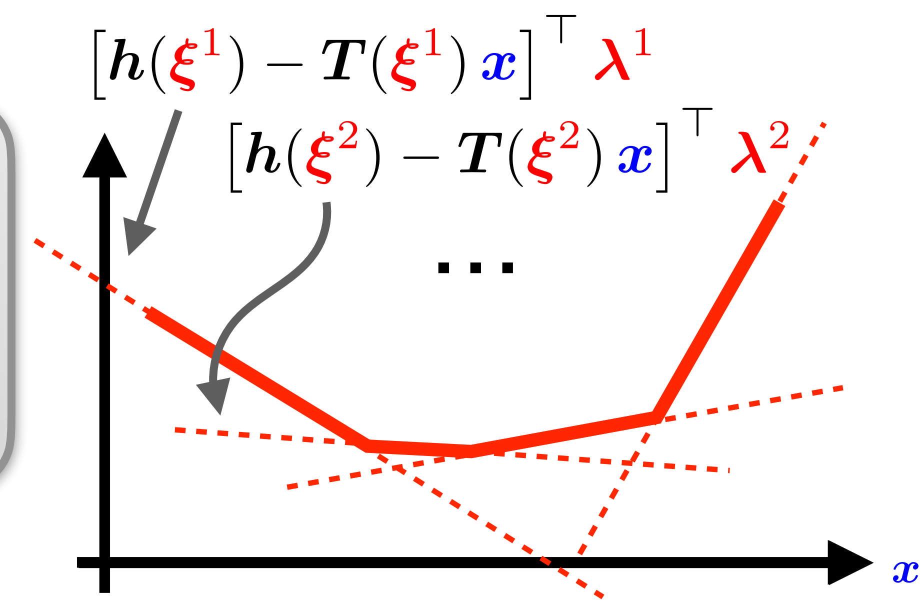
How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{ll} \underset{\boldsymbol{\xi},\boldsymbol{\lambda}}{\text{maximize}} & \left[\boldsymbol{h}(\boldsymbol{\xi}) - \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star\right]^\top \boldsymbol{\lambda} \\ \text{subject to} & \boldsymbol{W}^\top \boldsymbol{\lambda} \leq \boldsymbol{r}, \quad \boldsymbol{\xi} \in \Xi \end{array}$$

How do we incorporate these lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{x},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta \\ \text{subject to} & \theta \geq \left[\boldsymbol{h}(\boldsymbol{\xi}^i) - \boldsymbol{T}(\boldsymbol{\xi}^i)\,\boldsymbol{x}\right]^\top \boldsymbol{\lambda}^i \quad \forall i \in \mathcal{I} \\ & \boldsymbol{x} \in \mathcal{X} \end{array}$$



How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \, : \, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\iff \quad \begin{array}{ll} \underset{\boldsymbol{\xi},\boldsymbol{\lambda}}{\text{maximize}} & [\boldsymbol{h}(\boldsymbol{\xi}) - \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star]^\top \boldsymbol{\lambda} \\ \text{subject to} & \boldsymbol{W}^\top \boldsymbol{\lambda} \leq \boldsymbol{r}, \quad \boldsymbol{\xi} \in \Xi \end{array}$$

What if some first-stage decisions $\boldsymbol{x}^\star$ are infeasible in the second stage?

first solve the feasibility problem

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \underset{\boldsymbol{y},\boldsymbol{v}}{\min}\{\mathrm{e}^\top \boldsymbol{v} \, : \, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} + \boldsymbol{v} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

(always feasible by construction)

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

$$\Longleftrightarrow \quad \underbrace{\begin{array}{ll} \underset{\boldsymbol{\xi},\boldsymbol{\lambda}}{\text{maximize}} & [\boldsymbol{h}(\boldsymbol{\xi}) - \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star]^\top \boldsymbol{\lambda} \\ \text{subject to} & \boldsymbol{W}^\top \boldsymbol{\lambda} \leq \boldsymbol{r}, \quad \boldsymbol{\xi} \in \Xi \end{array}}$$

What if some first-stage decisions $\boldsymbol{x}^\star$ are infeasible in the second stage?

- first solve the feasibility problem (...)
- we iteratively generate:
  - feasibility cuts (approximating the first-stage feasible region)
  - optimality cuts (approximating $\mathcal{Q}(\boldsymbol{x})$)

Part 2 **Continuous Recourse Decisions**

Two-Stage Models

- Decision Rules
- Lower Bounds
- Benders' Decomposition
- **Column-and-Constraint Generation**
- Iterative Partitioning
- Fourier-Motzkin Elimination

Zheng and Zhao. *Solving Two-Stage Robust Optimization Problems using a Column-and-Constraint Generation Method*. ORL, 2013.

Column-and-Constraint Generation

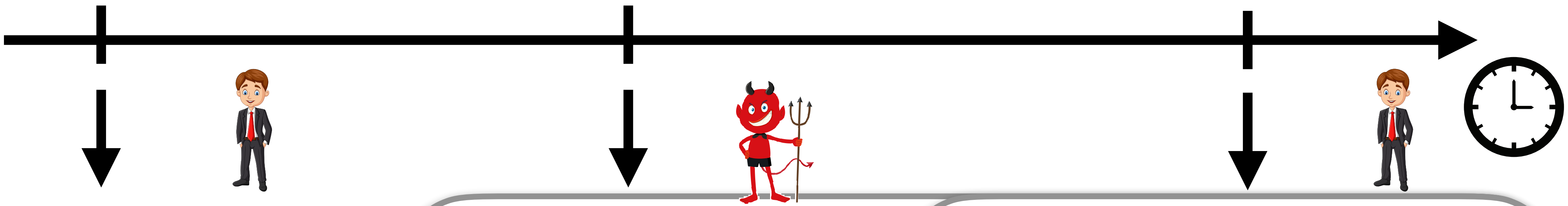
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Column-and-Constraint Generation

Recall the two-stage robust optimization problem:



$$\left[\begin{array}{ll}\min & c^\top x \\ \text{s.t.} & x \in \mathcal{X}\end{array}\right] + \left[\begin{array}{ll}\max & \cancel{q(\xi)^\top x} \\ \text{s.t.} & \xi \in \Xi\end{array}\right] + \left[\begin{array}{ll}\min & r^\top y \\ \text{s.t.} & y \in \mathcal{Y} \\ & T(\xi)\,x + W y \geq h(\xi)\end{array}\right]$$

$\mathcal{Q}(x)$

$Q(x;\xi)$

We can represent it as a single-stage optimization problem:

$$\begin{array}{ll}\underset{x}{\text{minimize}} & c^\top x + \mathcal{Q}(x) \\ \text{subject to} & x \in \mathcal{X}\end{array} \quad \text{with} \quad \mathcal{Q}(x) = \max_{\xi \in \Xi} \; Q(x;\xi)$$

$$Q(x;\xi) = \left[\begin{array}{ll}\underset{y}{\text{minimize}} & q^\top x + r^\top y \\ \text{subject to} & T(\xi)\,x + W y \geq h(\xi) \\ & y \in \mathcal{Y}\end{array}\right]$$

Column-and-Constraint Generation

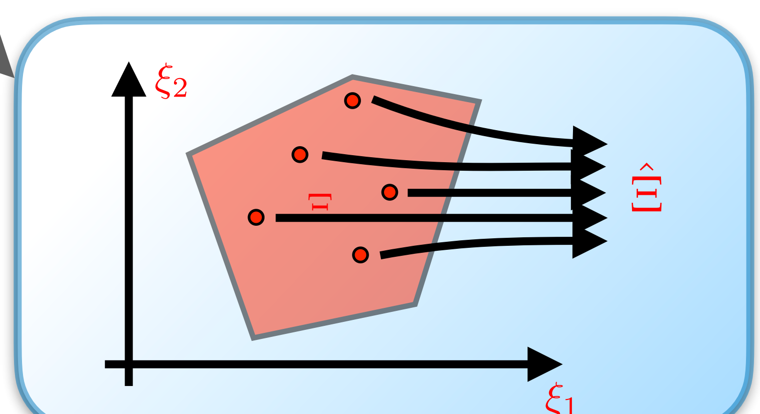
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Bound $\mathcal{Q}(\boldsymbol{x})$ from below through a finite scenario relaxation:

$$\underline{\mathcal{Q}}(\boldsymbol{x}) = \max_{\boxed{\boldsymbol{\xi} \in \hat{\Xi}}} \; Q(\boldsymbol{x}; \boldsymbol{\xi}) \quad \leq \quad \mathcal{Q}(\boldsymbol{x}) = \max_{\boldsymbol{\xi} \in \Xi} \; Q(\boldsymbol{x}; \boldsymbol{\xi})$$



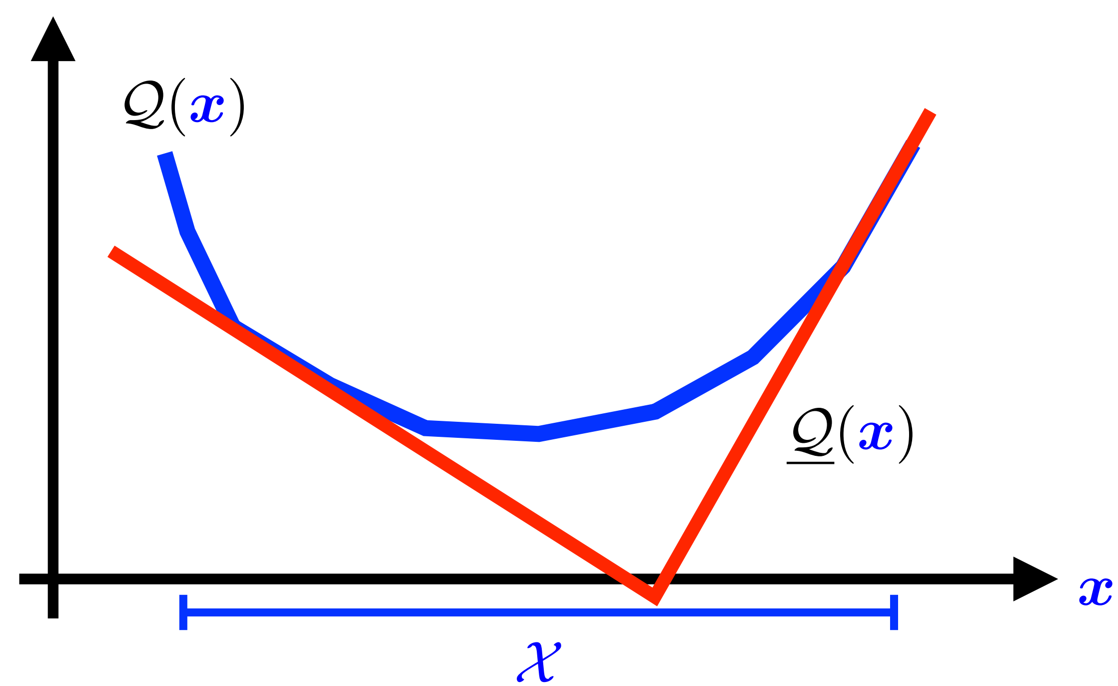
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \; + \; \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \; + \; \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

The finite scenario approximation tends to converge faster:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



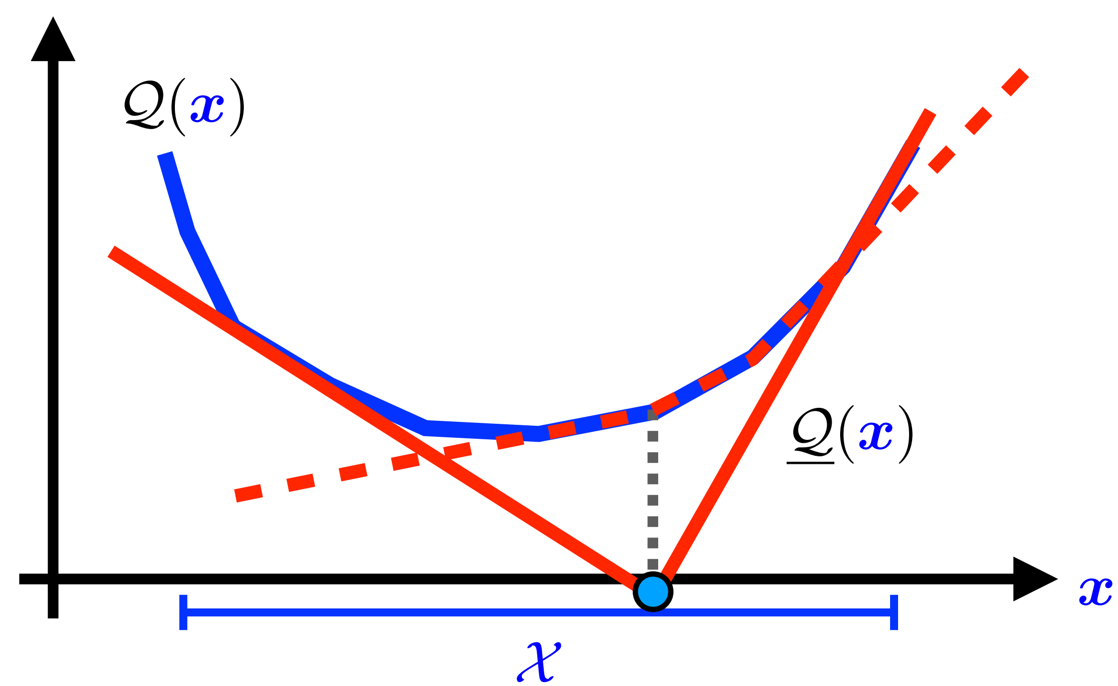
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

The finite scenario approximation tends to converge faster:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



Column-and-Constraint Generation

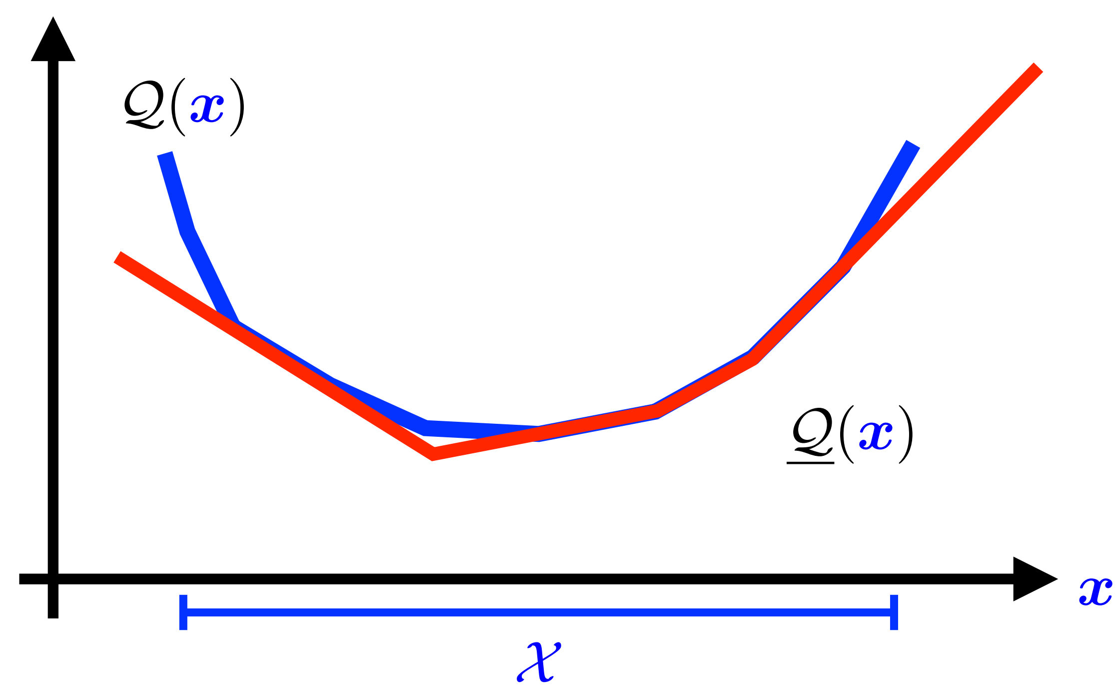
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top} \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

The finite scenario approximation tends to converge faster:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



Column-and-Constraint Generation

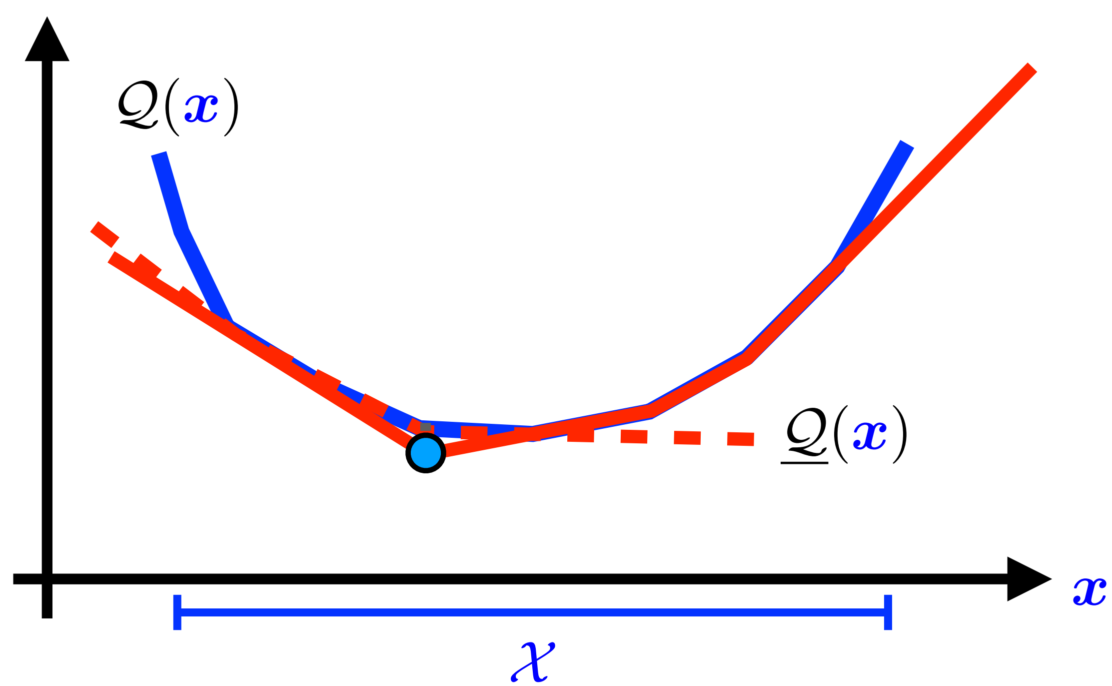
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

The finite scenario approximation tends to converge faster:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



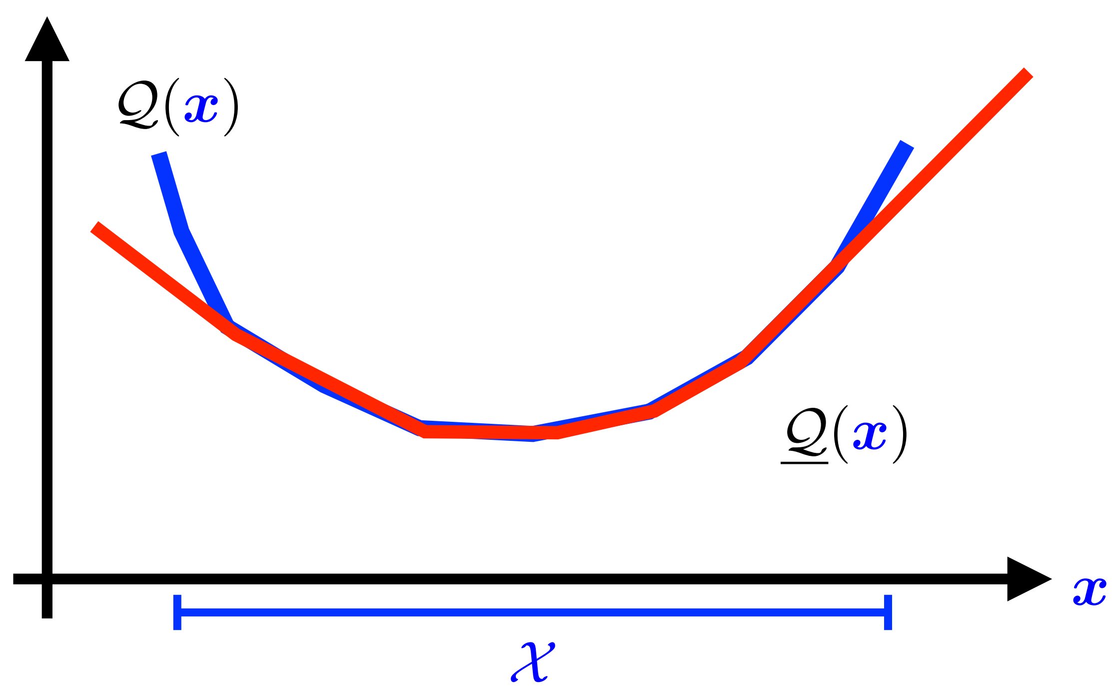
Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

The finite scenario approximation tends to converge faster:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X} \end{array}$$



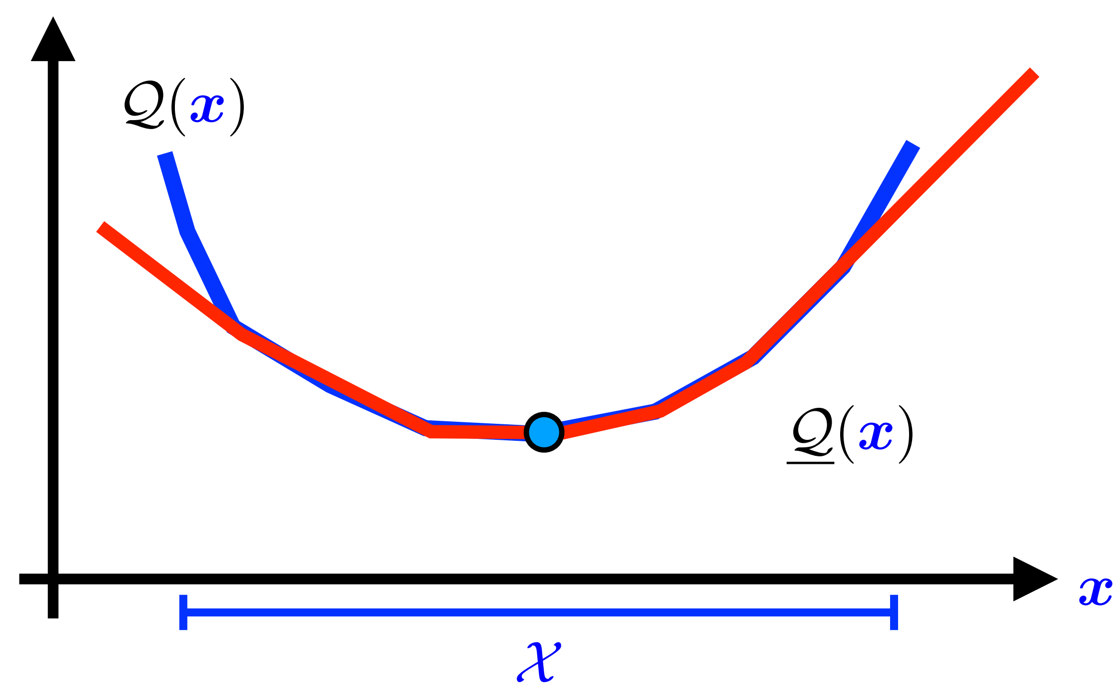
Recall the two-stage robust optimization problem:

$$\begin{array}{ll}\min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X}\end{array} \quad + \quad \begin{array}{ll}\max & \cancel{\boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x}} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi\end{array} \quad + \quad \begin{array}{ll}\min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y}\end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

The finite scenario approximation tends to converge faster:

$$\begin{array}{ll}\underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \mathcal{Q}(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}\end{array}$$



How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

hard problem: solution via global optimization solvers, MILP reformulations (via KKT conditions or for structured uncertainty sets Ξ)

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

- **hard problem:** solution via global optimization solvers, MILP reformulations (via KKT conditions or for structured uncertainty sets Ξ)

- $\boldsymbol{x} \mapsto \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi}^\star)\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi}^\star)\}$ **supports** $\mathcal{Q}(\boldsymbol{x})$ **at** $\boldsymbol{x} = \boldsymbol{x}^\star$

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

- **hard problem:** solution via global optimization solvers, MILP reformulations (via KKT conditions or for structured uncertainty sets Ξ)
- $\boldsymbol{x} \mapsto \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi}^\star)\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi}^\star)\}$ **supports** $\mathcal{Q}(\boldsymbol{x})$ **at** $\boldsymbol{x} = \boldsymbol{x}^\star$
- $\boldsymbol{x} \mapsto \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi}^\star)\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi}^\star)\}$ **bounds** $\mathcal{Q}(\boldsymbol{x})$ **from below:** since $\boldsymbol{\xi}^\star$ is a feasible, but—when $\boldsymbol{x} \neq \boldsymbol{x}^\star$—typically suboptimal choice for the adversary

Column-and-Constraint Generation

How can we construct the lower bounds?

$$\begin{array}{ll} \underset{\boldsymbol{\xi}}{\text{maximize}} & \min\{\boldsymbol{r}^\top \boldsymbol{y} \,:\, \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})\} \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \end{array}$$

How do we incorporate these lower bounds?

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}^i) & \forall i \in \mathcal{I} \\ & \boldsymbol{T}(\boldsymbol{\xi}^i)\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}^i) \geq \boldsymbol{h}(\boldsymbol{\xi}^i) & \forall i \in \mathcal{I} \\ & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}(\boldsymbol{\xi}^i) \in \mathcal{Y}, \, i \in \mathcal{I} & \end{array}$$

similar to the Hadjiyiannis scenario relaxation!

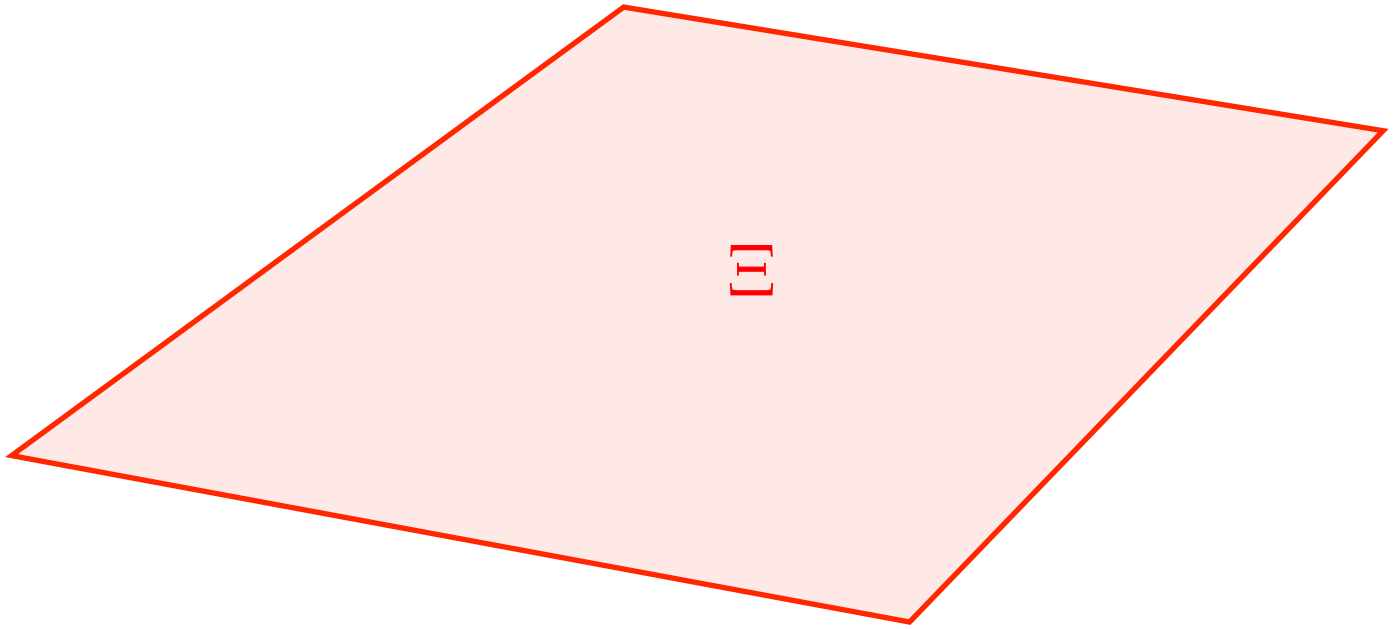
Part 2 **Continuous Recourse Decisions**

Two-Stage Models

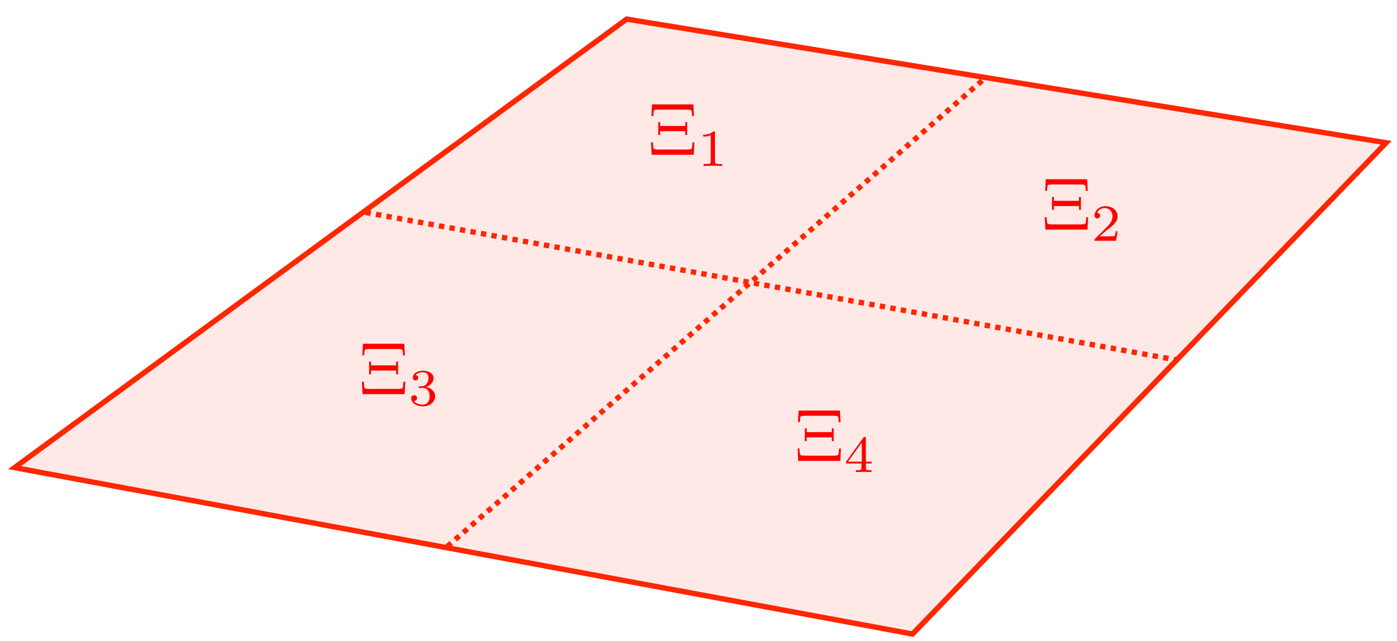
- Decision Rules
- Lower Bounds
- Benders' Decomposition
- Column-and-Constraint Generation
- **Iterative Partitioning**
- Fourier-Motzkin Elimination

Bertsimas and Dunning. *Multistage Robust Mixed-Integer Optimization with Adaptive Partitions*. OR, 2016.
Postek and den Hertog. *Multistage Adjustable Robust Mixed-Integer Optimization via Iterative Splitting of the Uncertainty Set*. INFORMS JoC, 2016.

Idea: *manually* split uncertainty set into subsets and assign individual decision rules to each subset



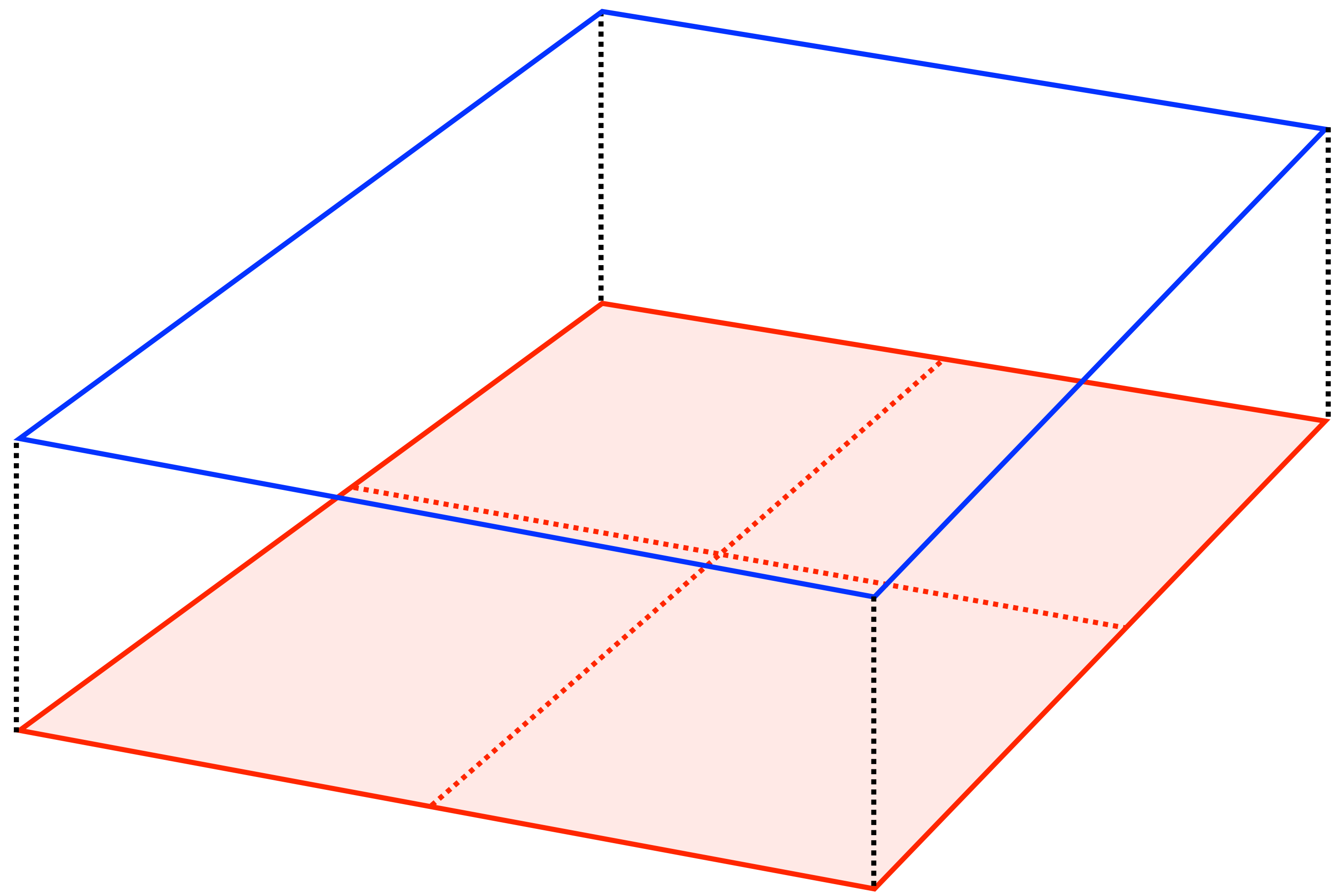
Idea: *manually* split uncertainty set into subsets and assign individual decision rules to each subset



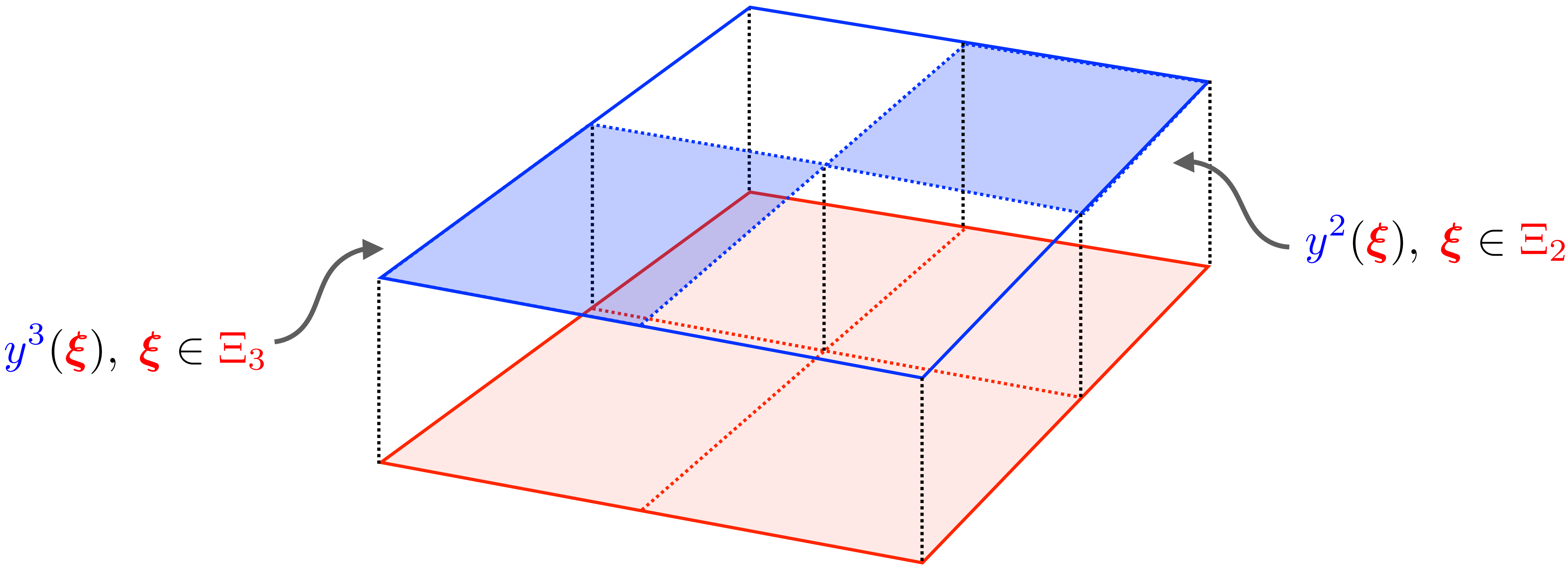
choose a partition of Ξ into $\Xi_1, \ldots, \Xi_4$ such that

$$\Xi_1 \cup \Xi_2 \cup \Xi_3 \cup \Xi_4 \stackrel{!}{=} \Xi$$

Idea: *manually* split uncertainty set into subsets and assign individual decision rules to each subset

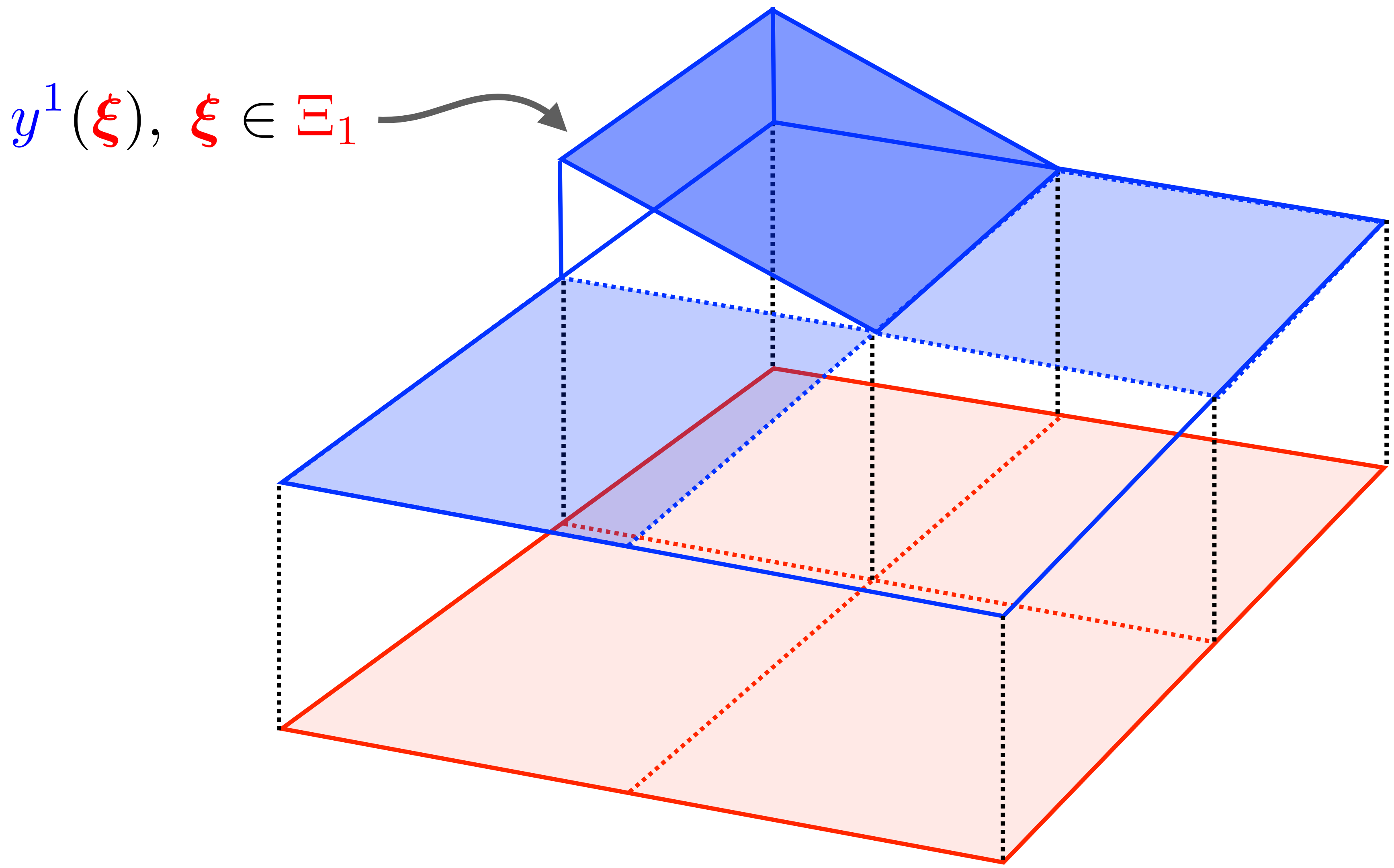


Idea: *manually* split uncertainty set into subsets and assign individual decision rules to each subset



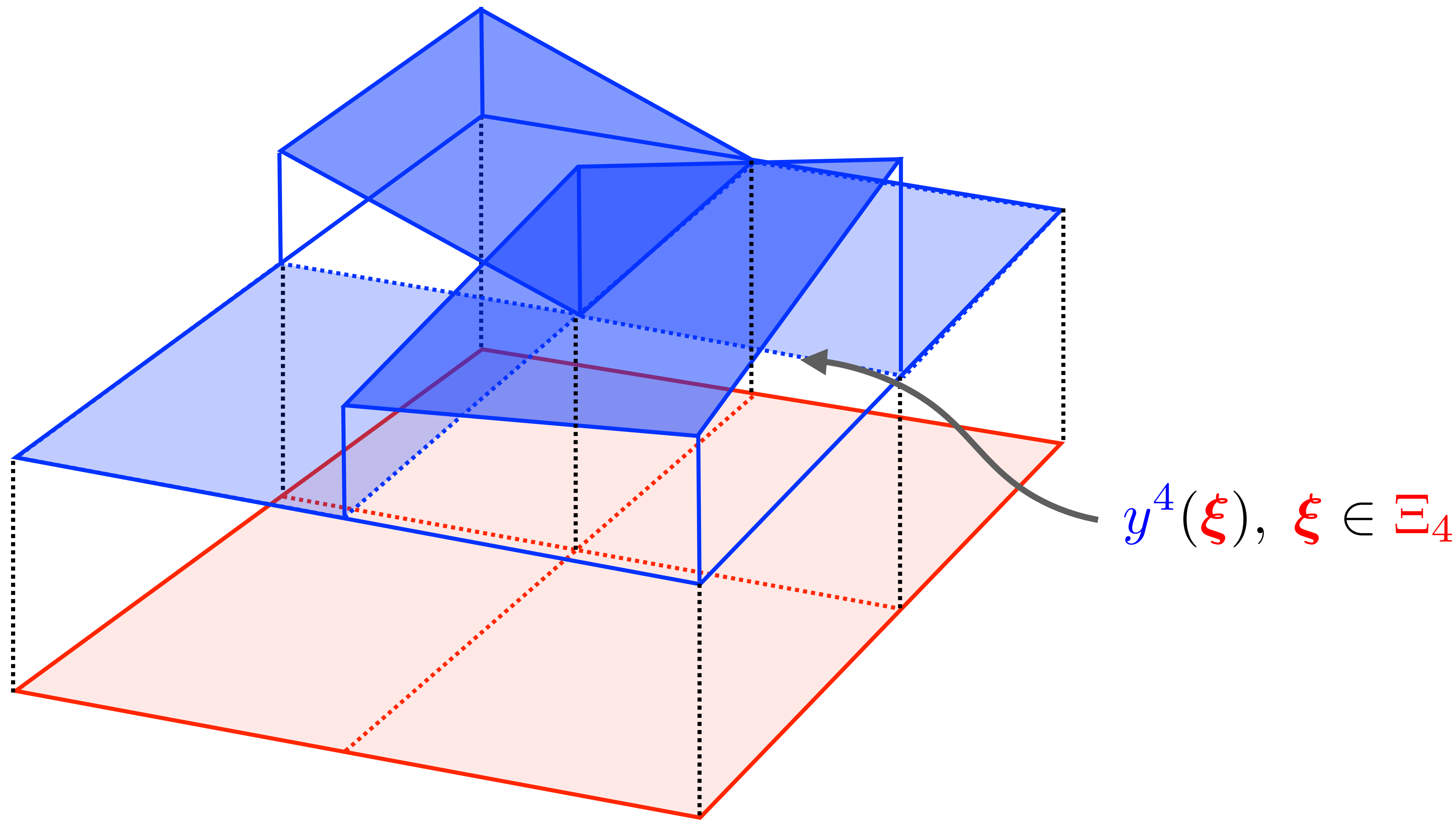
assign a different decision rule $y^i(\boldsymbol{\xi})$ to each subset $i = 1, \ldots, 4$ of the partition

Idea: *manually* split uncertainty set into subsets and assign individual decision rules to each subset



assign a different decision rule $y^i(\boldsymbol{\xi})$ to each subset $i = 1, \ldots, 4$ of the partition

Idea: *manually* split uncertainty set into subsets and assign individual decision rules to each subset



assign a different decision rule $y^i(\boldsymbol{\xi})$ to each subset $i = 1, \ldots, 4$ of the partition

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\,\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

$$\Downarrow$$

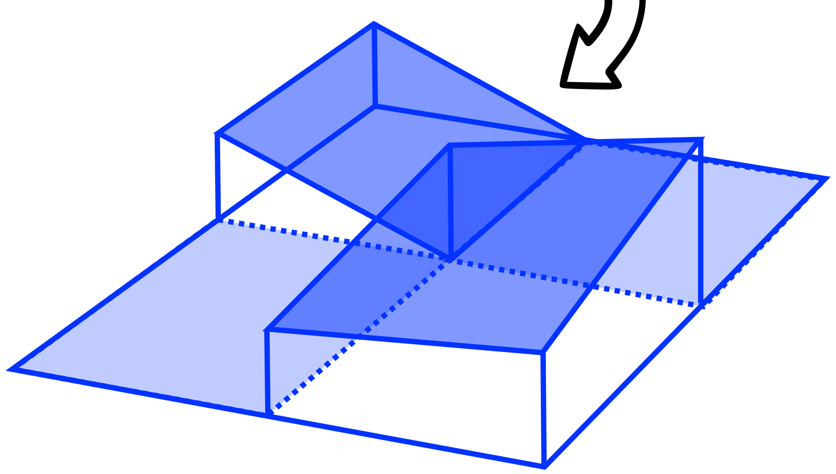
$$\begin{array}{ll} \underset{\boldsymbol{x},\{\boldsymbol{y}^i\}_i}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^i(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi} \in \Xi^i, \ \forall i \in \mathcal{I} \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^i : \Xi^i \mapsto \mathcal{Y}, \ i \in \mathcal{I} \end{array}$$

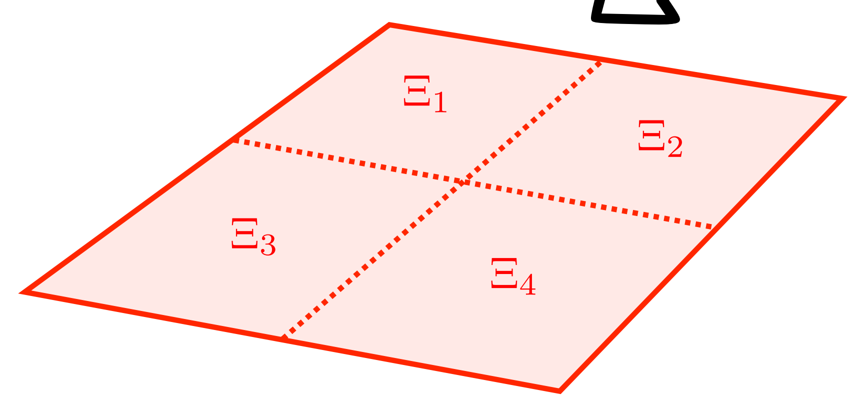
Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} & \end{array}$$

$$\Downarrow$$

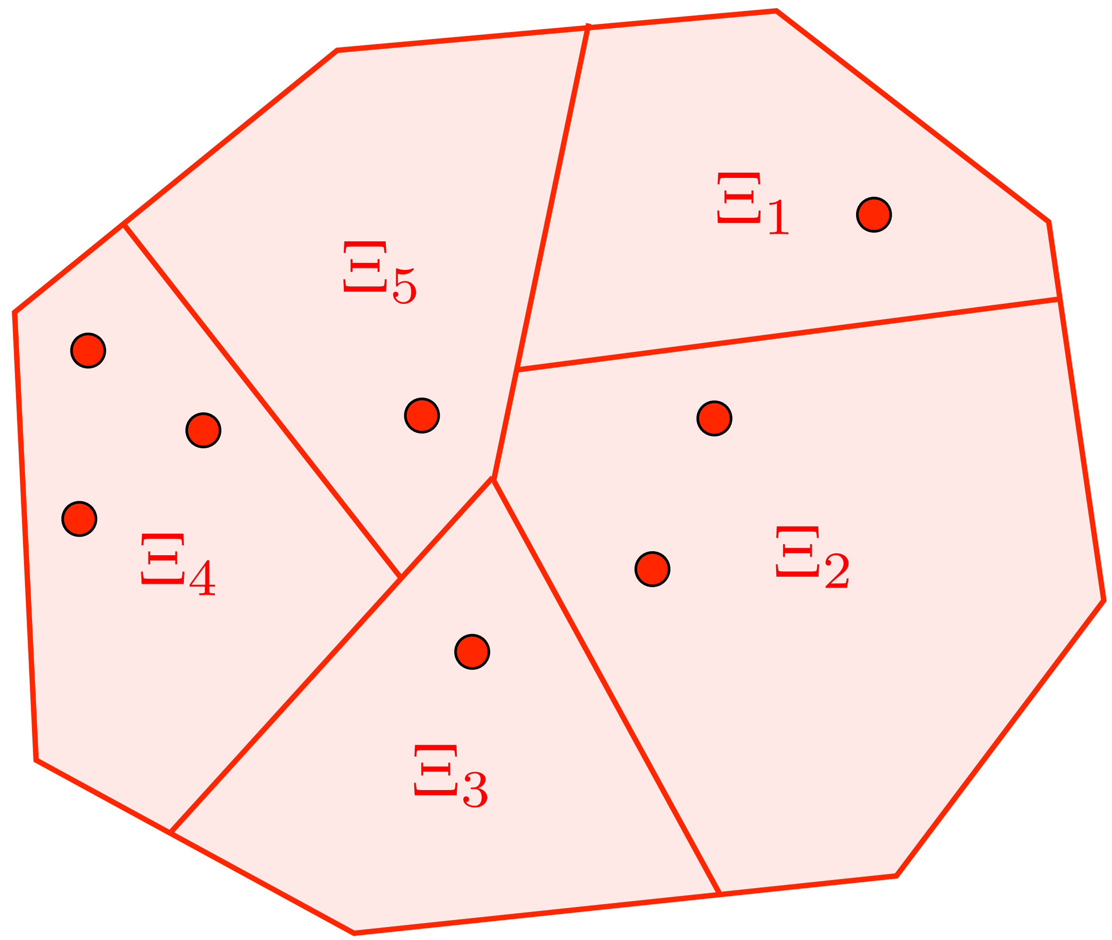
$$\begin{array}{lll} \underset{\boldsymbol{x},\{\boldsymbol{y}^i\}_i}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^i(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \boxed{\forall \boldsymbol{\xi} \in \Xi^i, \ \forall i \in \mathcal{I}} \\ & \boldsymbol{x} \in \mathcal{X}, \ \boxed{\boldsymbol{y}^i} : \Xi^i \mapsto \mathcal{Y}, \boxed{i \in \mathcal{I}} & \end{array}$$





Iterative Partitioning

Imagine a two-stage RO problem with the uncertainty set:



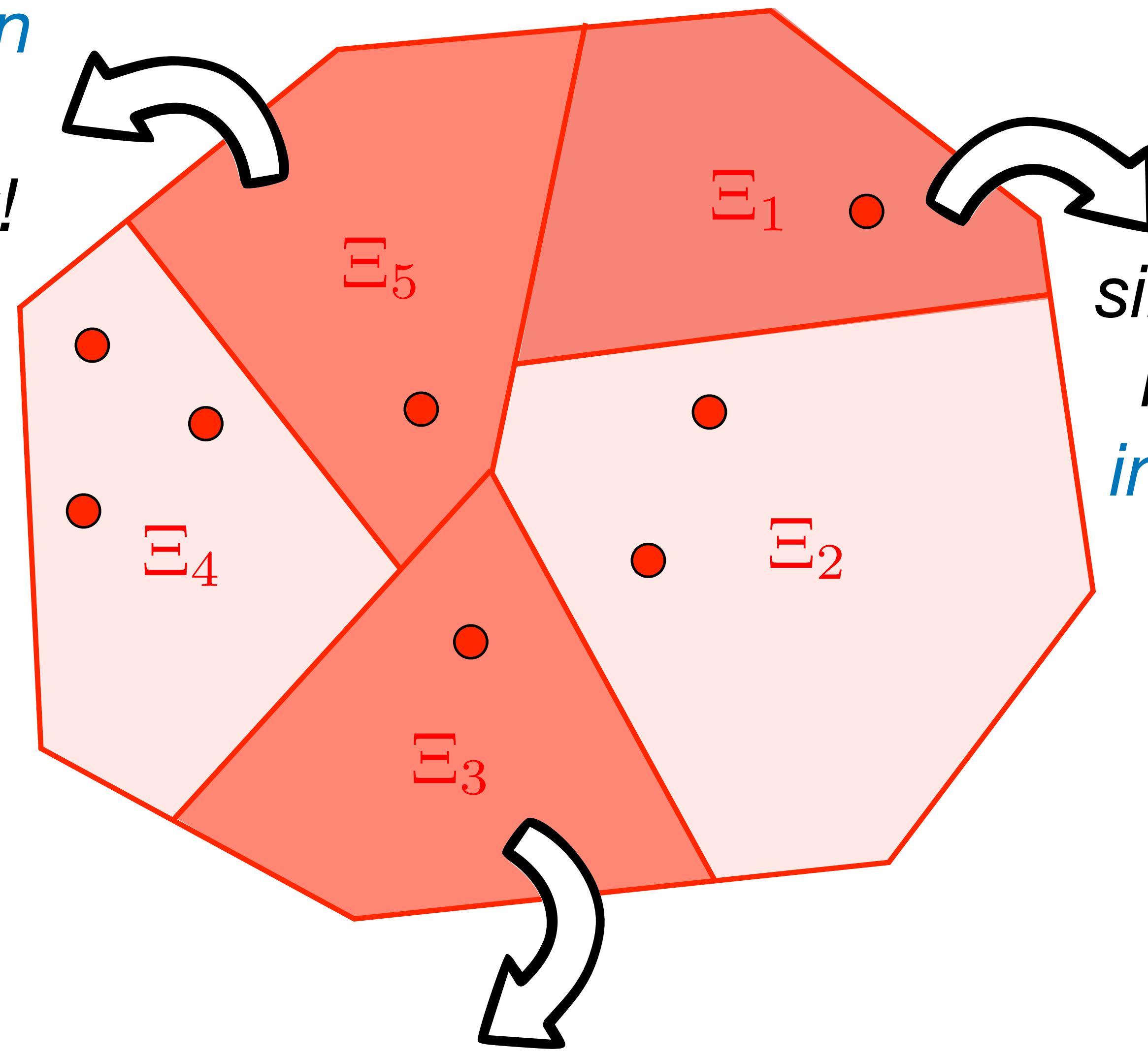
● = *scenarios* that are *binding* *for at least one of the* *constraints* *in the problem*

active (or *critical*) *scenarios*

Iterative Partitioning

Imagine a two-stage RO problem with the uncertainty set:

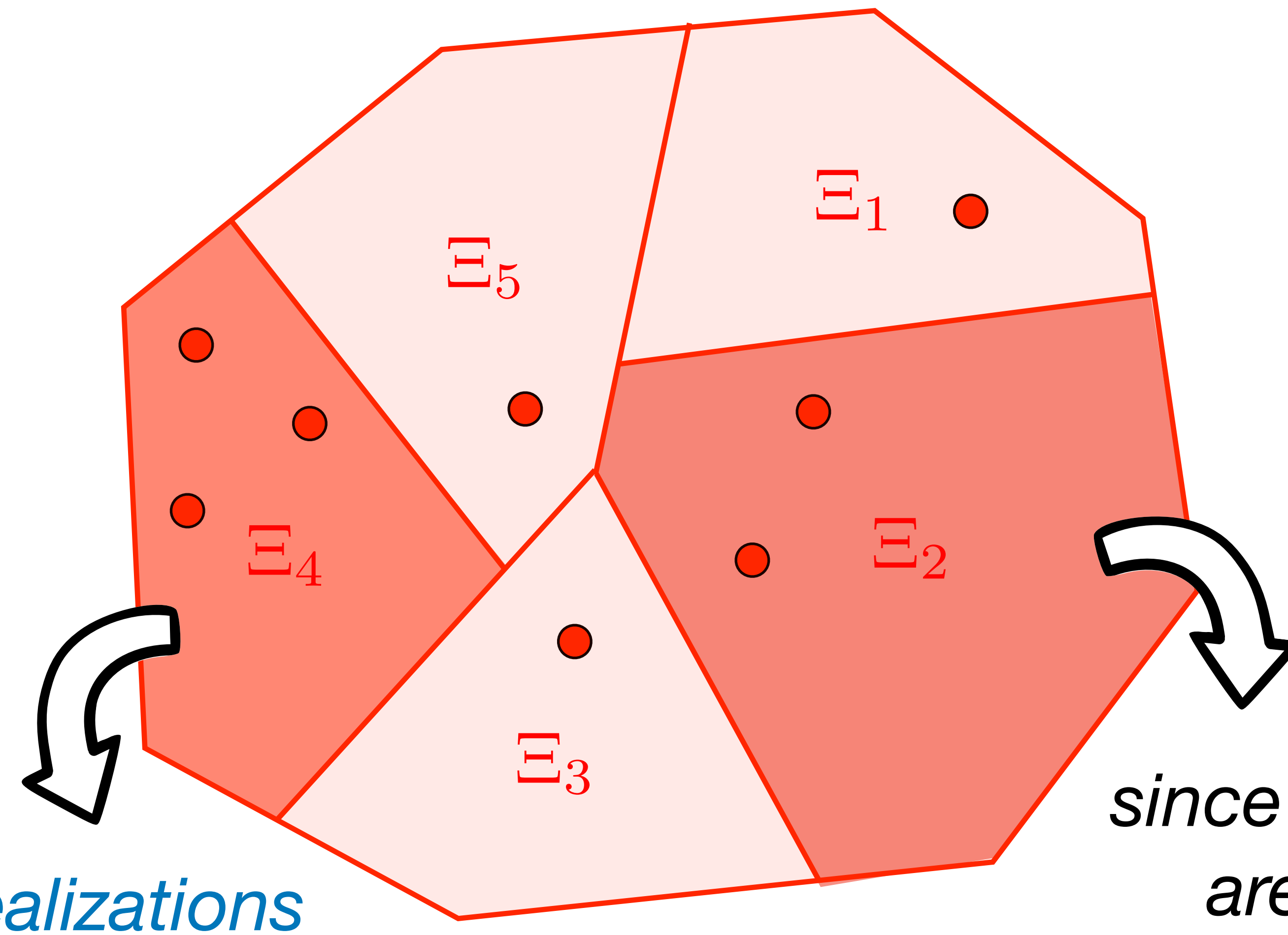
since a single realization is binding, we cannot improve on this subset!



since a single realization is binding, we cannot improve on this subset!

since a single realization is binding, we cannot improve on this subset!

Imagine a two-stage RO problem with the uncertainty set:



since multiple realizations are binding, we can potentially improve through the right splits!

since multiple realizations are binding, we can potentially improve through the right splits!

Why is that?

Why is that?

1

since a single realization is binding,
we cannot improve on this subset!

Why is that?

1

since a single realization is binding,
we cannot improve on this subset!

- **Assume that for fixed $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$, the constraint set**

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y}^\star(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi_i$$

is only binding at the uncertainty realization $\xi^\star \in \Xi_i$.

Why is that?

1

since a single realization is binding,
we cannot improve on this subset!

- **Assume that for fixed $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$, the constraint set**

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y}^\star(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi_i$$

is only binding at the uncertainty realization $\boldsymbol{\xi}^\star \in \Xi_i$.

- **Then convexity allows us to equivalently write the constraint set as**

$$\boldsymbol{T}(\boldsymbol{\xi}^\star)\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y}^\star(\boldsymbol{\xi}^\star) \geq \boldsymbol{h}(\boldsymbol{\xi}^\star)$$

Why is that?

1

since a single realization is binding, we cannot improve on this subset!

- **Assume that for fixed $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$, the constraint set**

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y}^\star(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi_i$$

  is only binding at the uncertainty realization $\boldsymbol{\xi}^\star \in \Xi_i$.

- **Then convexity allows us to equivalently write the constraint set as**

$$\boldsymbol{T}(\boldsymbol{\xi}^\star)\,\boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y}^\star(\boldsymbol{\xi}^\star) \geq \boldsymbol{h}(\boldsymbol{\xi}^\star)$$

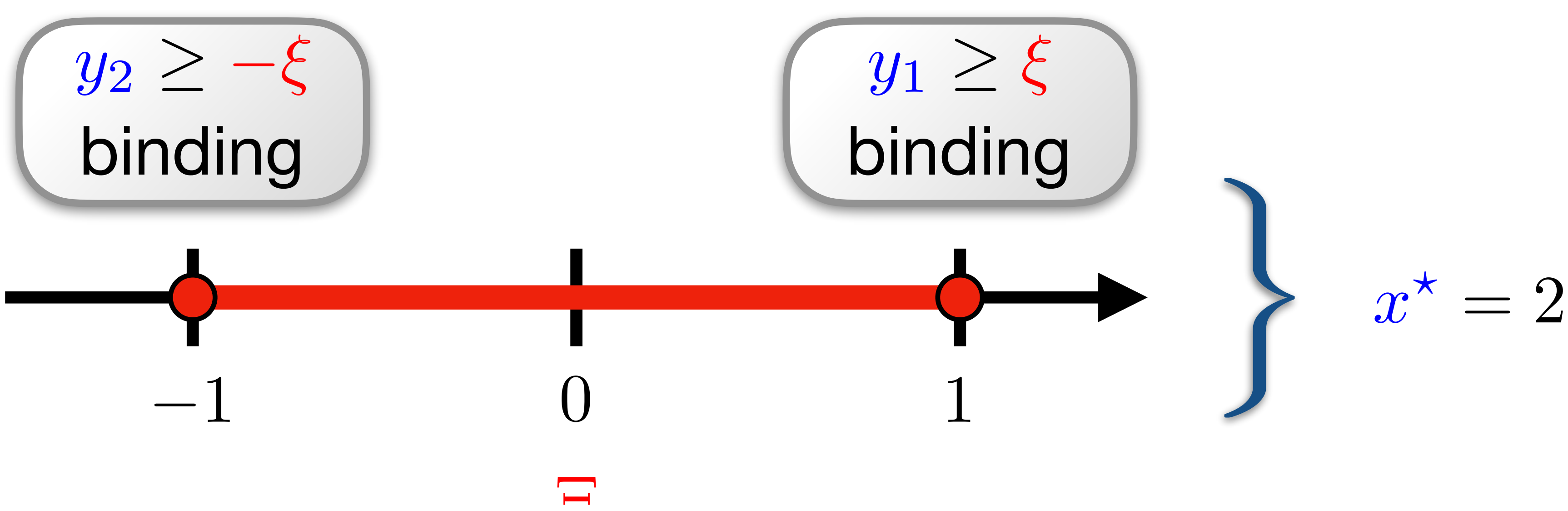
- **For constant/affine decision rules, this implies that the decision must be optimal!**

Why is that?

2 *since multiple realizations are binding, we can potentially improve through the right splits!*

Example:

$$
\begin{array}{lll}
\underset{x,\boldsymbol{y}}{\text{minimize}} & x & \\
\text{subject to} & x \geq y_1 + y_2 & \forall \xi \in [-1,1] \\
& y_1 \geq \xi & \forall \xi \in [-1,1] \\
& y_2 \geq -\xi & \forall \xi \in [-1,1] \\
& x, y_1, y_2 \in \mathbb{R} &
\end{array}
$$

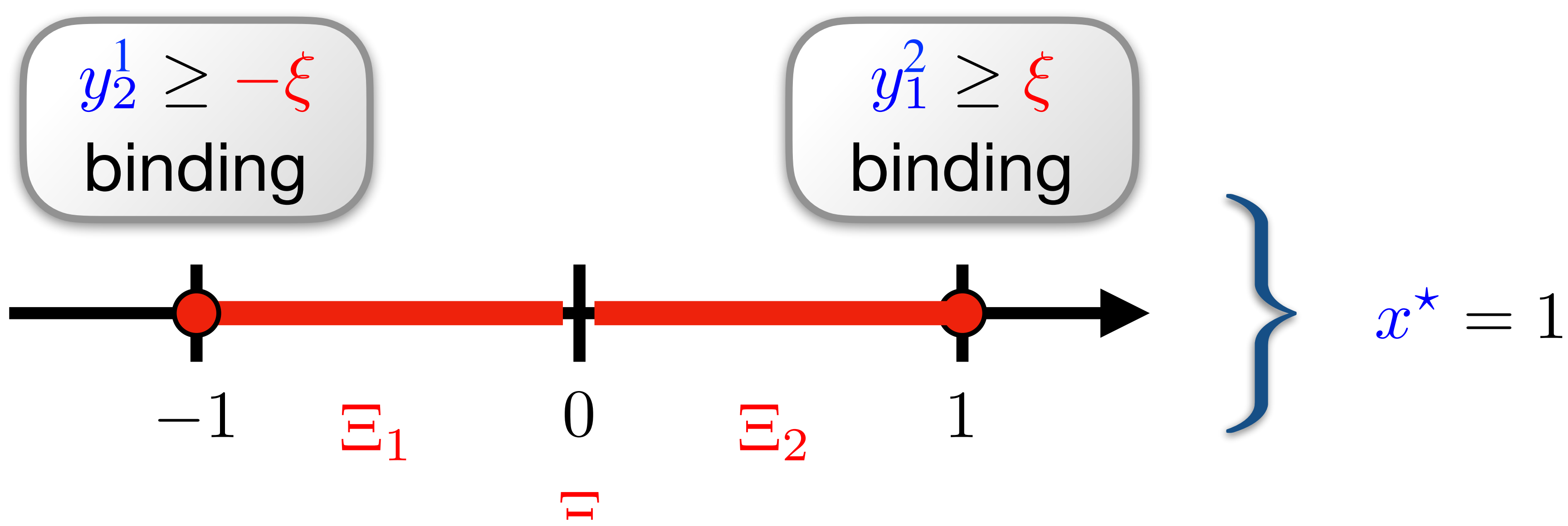


Why is that?

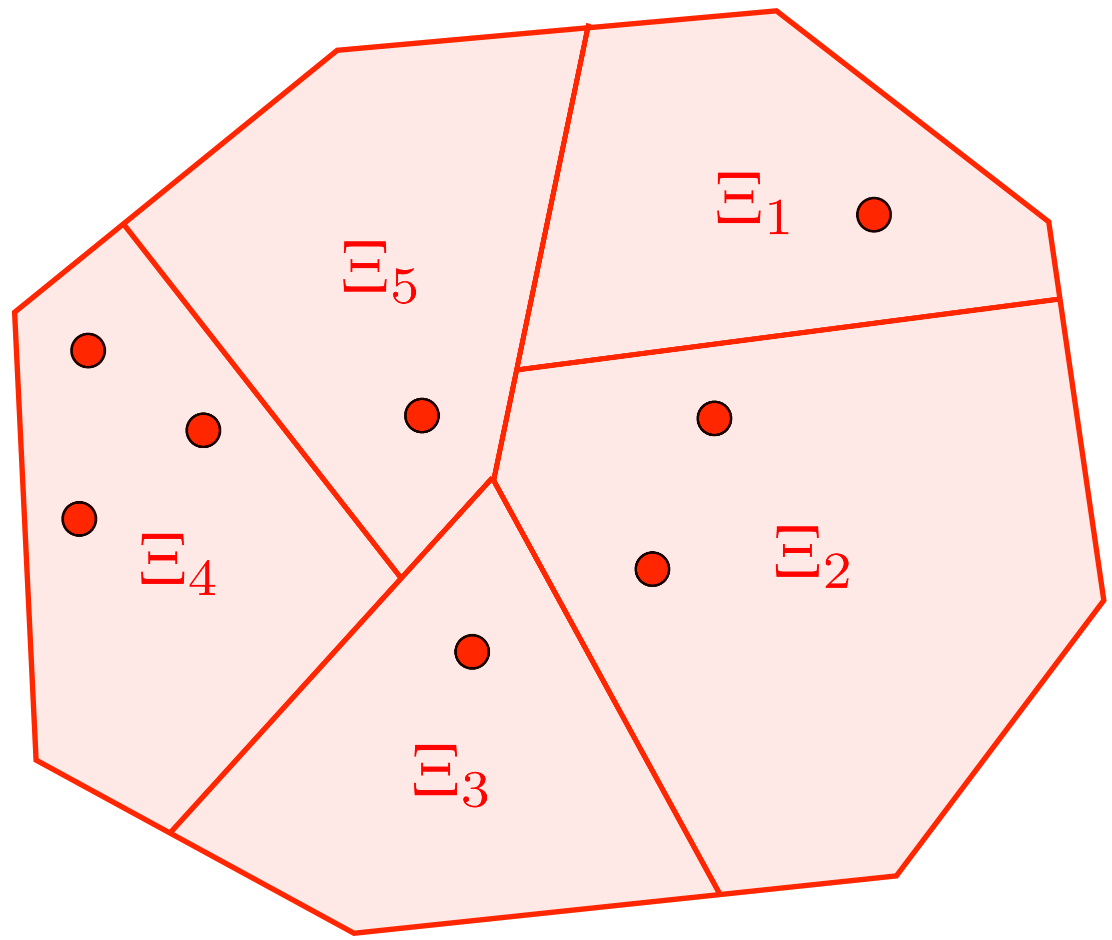
2 *since multiple realizations are binding, we can potentially improve through the right splits!*

Example:

$$
\begin{array}{lllll}
\underset{x,\boldsymbol{y}}{\text{minimize}} & x & & & \\
\text{subject to} & x \geq y_1^1 + y_2^1 & \forall \xi \in [-1,0], & x \geq y_1^2 + y_2^2 & \forall \xi \in [0,1] \\
& y_1^1 \geq \xi & \forall \xi \in [-1,0], & y_1^2 \geq \xi & \forall \xi \in [0,1] \\
& y_2^1 \geq -\xi & \forall \xi \in [-1,0], & y_2^2 \geq -\xi & \forall \xi \in [0,1] \\
& x, y_1^i, y_2^i \in \mathbb{R} & & &
\end{array}
$$



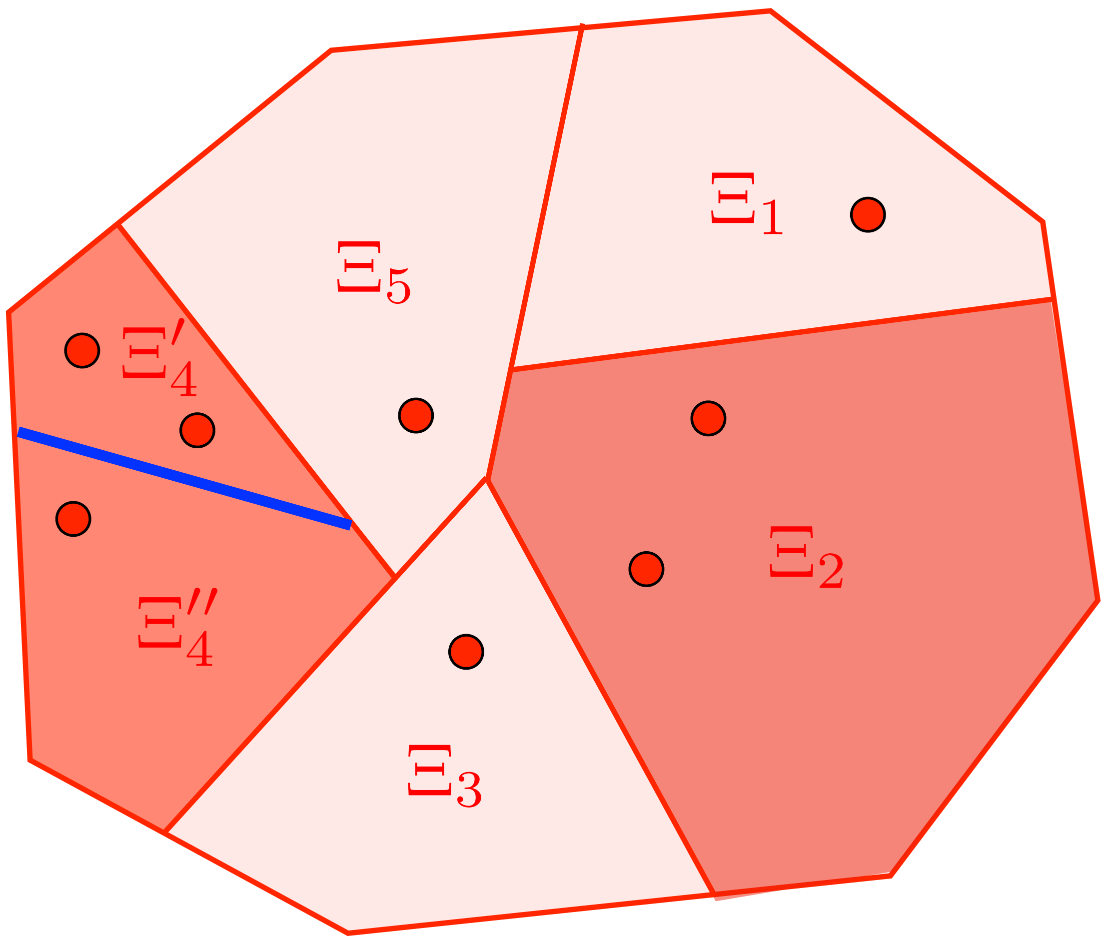
Imagine a two-stage RO problem with the uncertainty set:



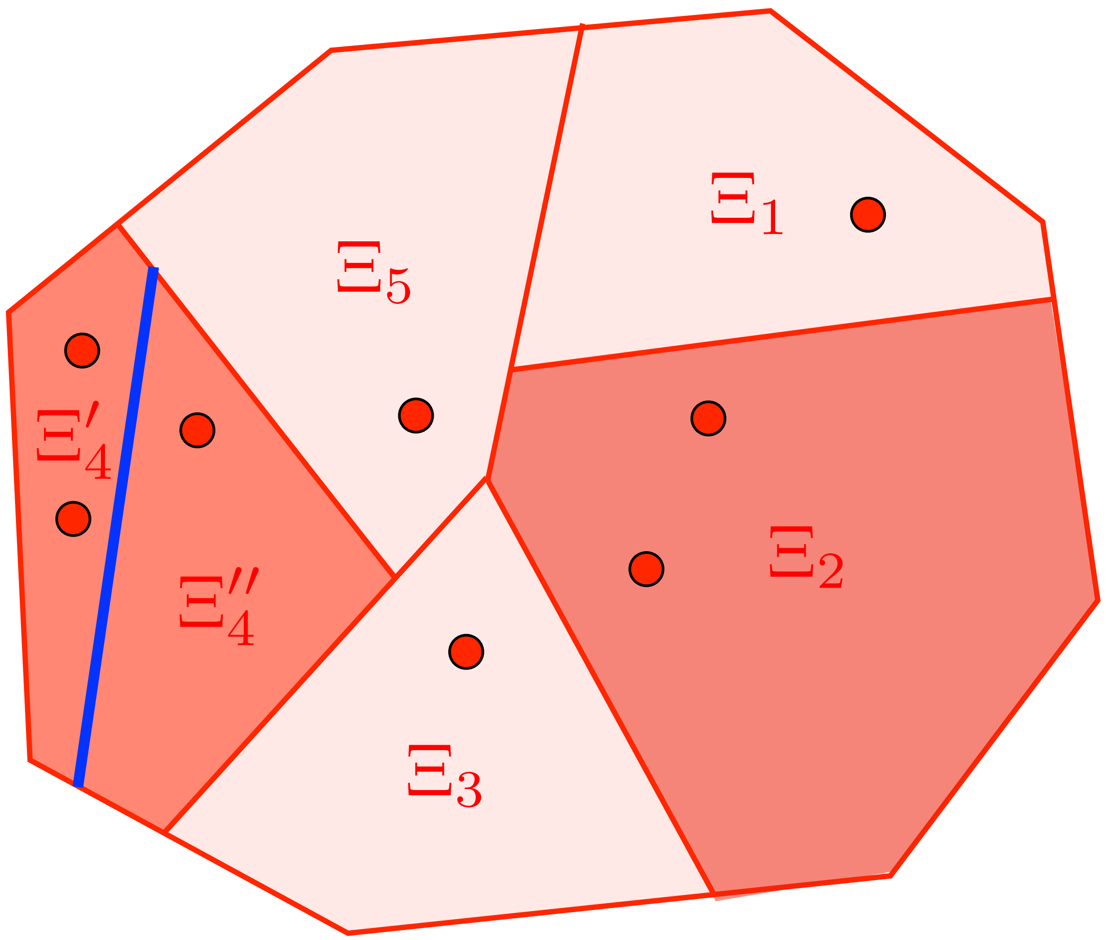
Necessary Condition for Improvement

If a subset Ξ_i is split into Ξ_i' and Ξ_i'' such that all active (critical) realizations $\boldsymbol{\xi} \in \Xi_i$ are contained in Ξ_i' (or Ξ_i''), then the optimal value of the two-stage RO cannot improve.

Imagine a two-stage RO problem with the uncertainty set:



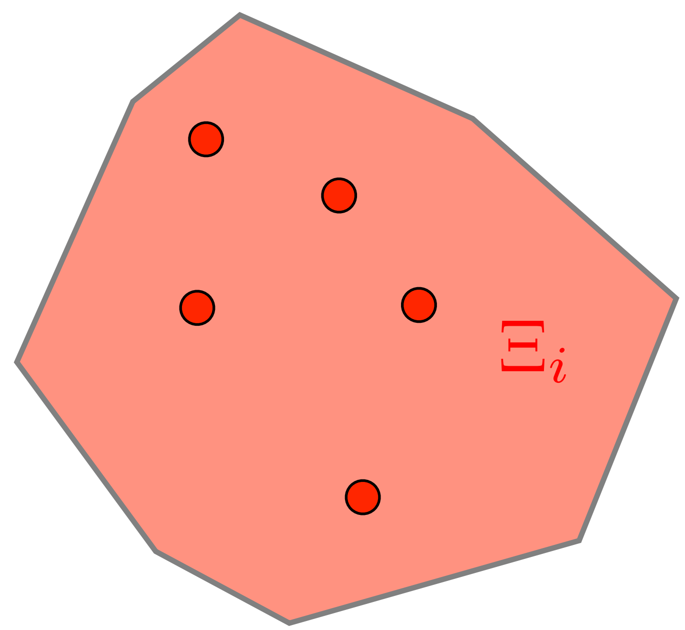
Imagine a two-stage RO problem with the uncertainty set:



Iterative Partitioning: Splitting Heuristics

HEURISTIC 1:

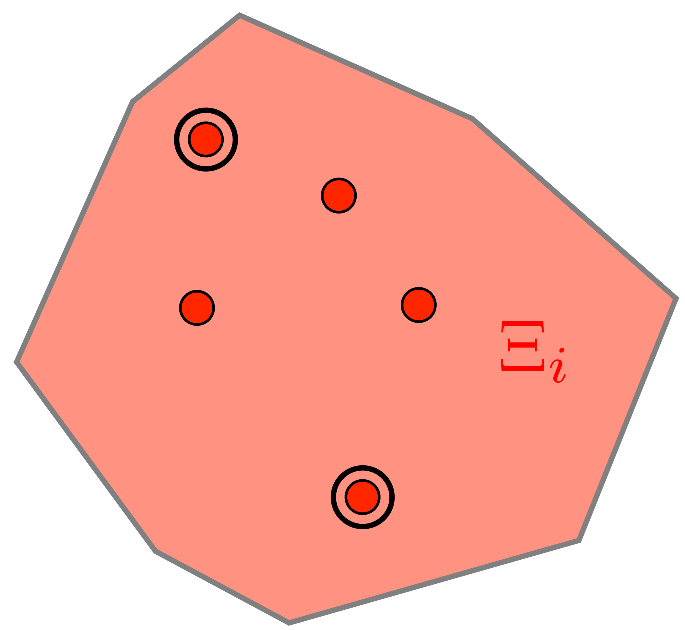
separate the farthest two scenarios "mid-way"



Iterative Partitioning: Splitting Heuristics

HEURISTIC 1:

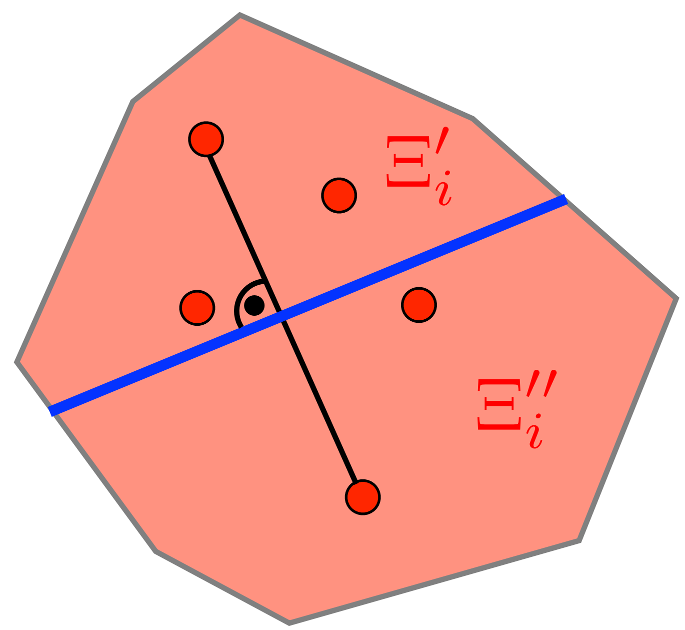
separate the farthest two scenarios “mid-way”



Iterative Partitioning: Splitting Heuristics

HEURISTIC 1:

separate the farthest two scenarios "mid-way"



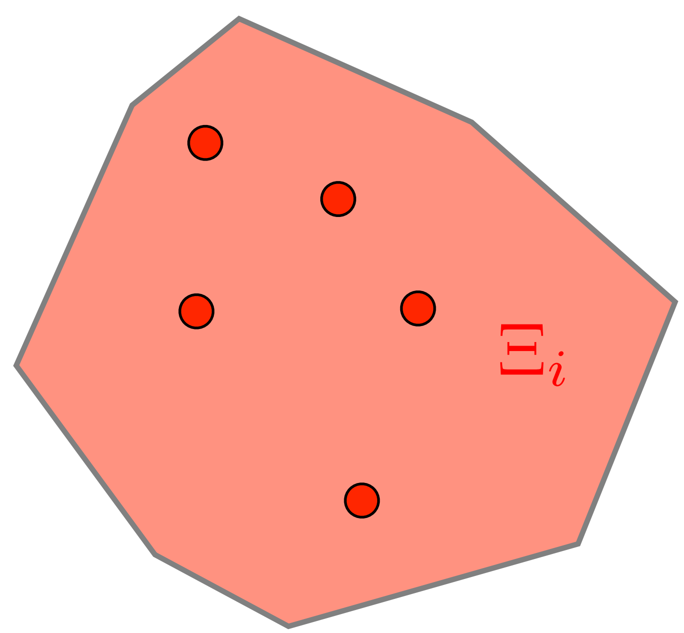
Iterative Partitioning: Splitting Heuristics

HEURISTIC 1:

separate the farthest two scenarios "mid-way"

HEURISTIC 2:

split subset as evenly as possible into two subsets



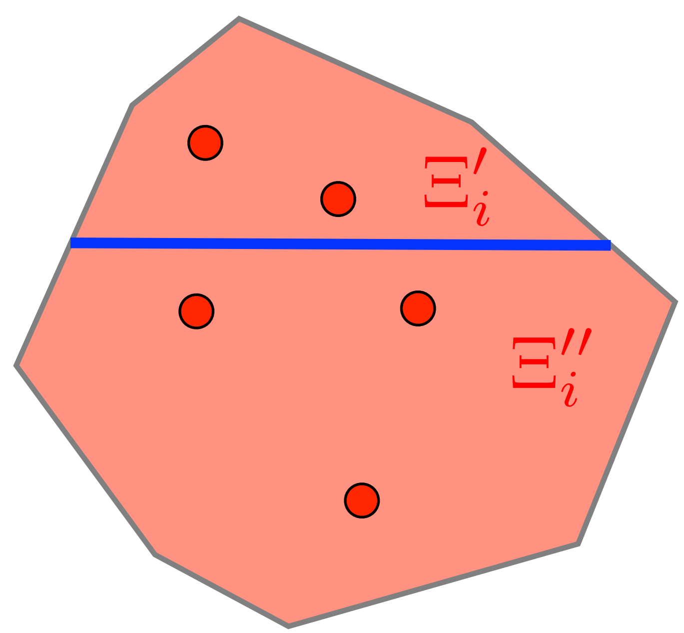
Iterative Partitioning: Splitting Heuristics

HEURISTIC 1:

separate the farthest two scenarios "mid-way"

HEURISTIC 2:

split subset as evenly as possible into two subsets



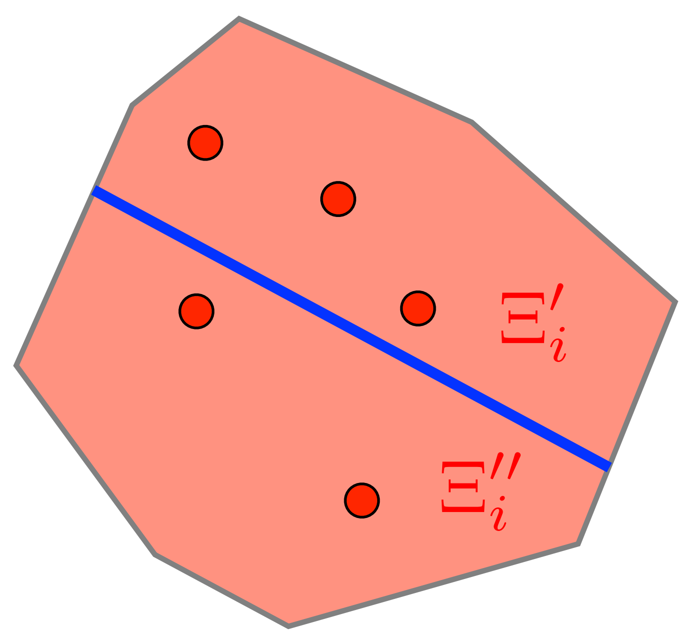
Iterative Partitioning: Splitting Heuristics

HEURISTIC 1:

separate the farthest two scenarios "mid-way"

HEURISTIC 2:

split subset as evenly as possible into two subsets



Iterative Partitioning: Splitting Heuristics

HEURISTIC 1:

separate the farthest two scenarios "mid-way"

HEURISTIC 2:

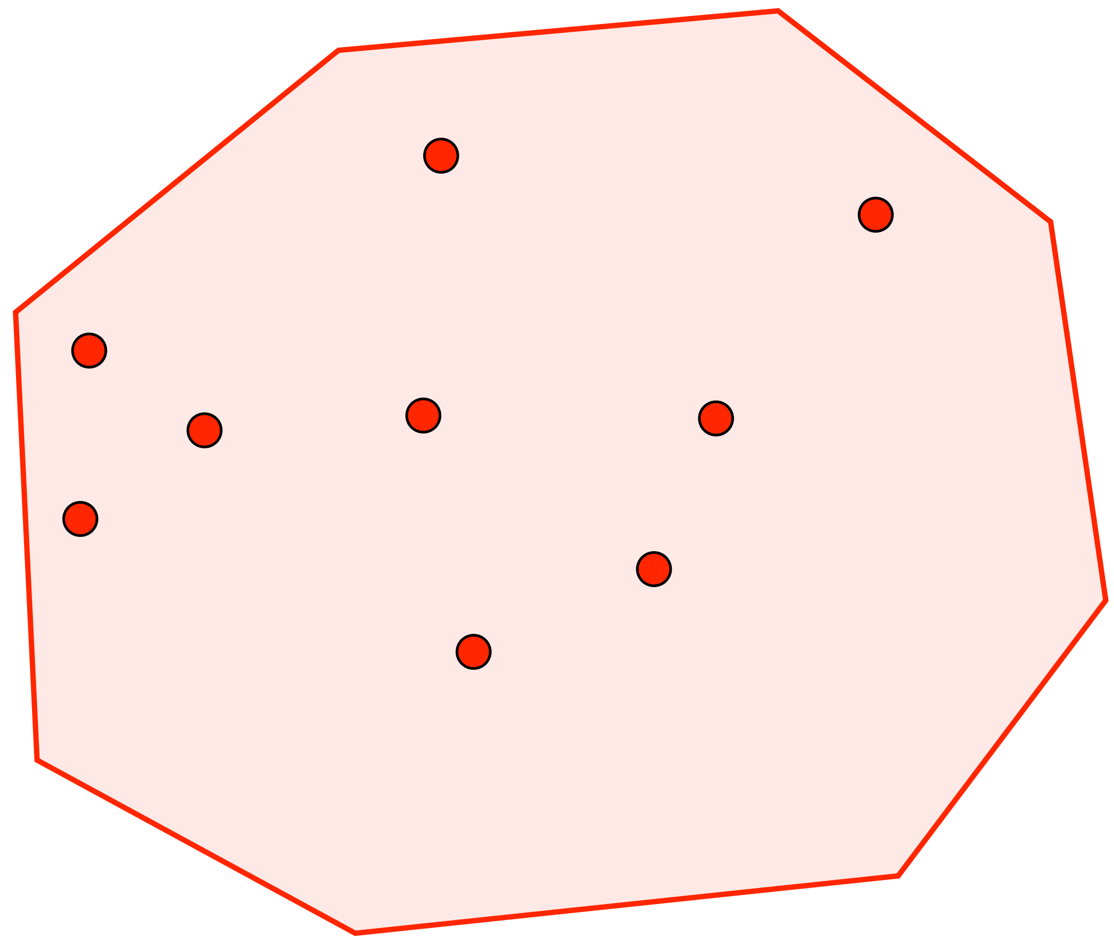
split subset as evenly as possible into two subsets

HEURISTIC 3:

minimize larger worst case over the two subsets

Iterative Partitioning: Voronoi Partitioning

Imagine a two-stage RO problem with the uncertainty set:

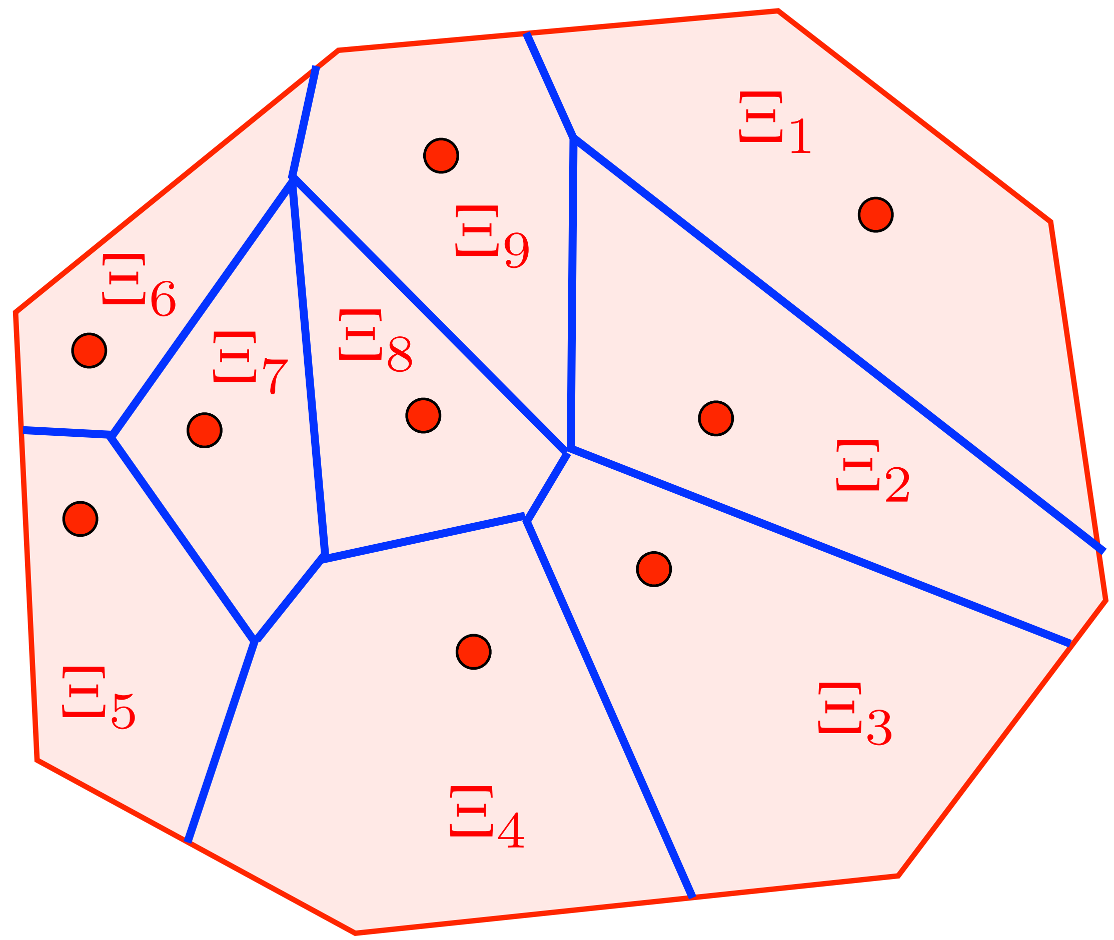


dots = scenarios that are binding for at least one of the constraints in the problem

active (or critical) scenarios

Iterative Partitioning: Voronoi Partitioning

We can partition the uncertainty set into a Voronoi diagram:



Georgy Feodosevich Voronoy

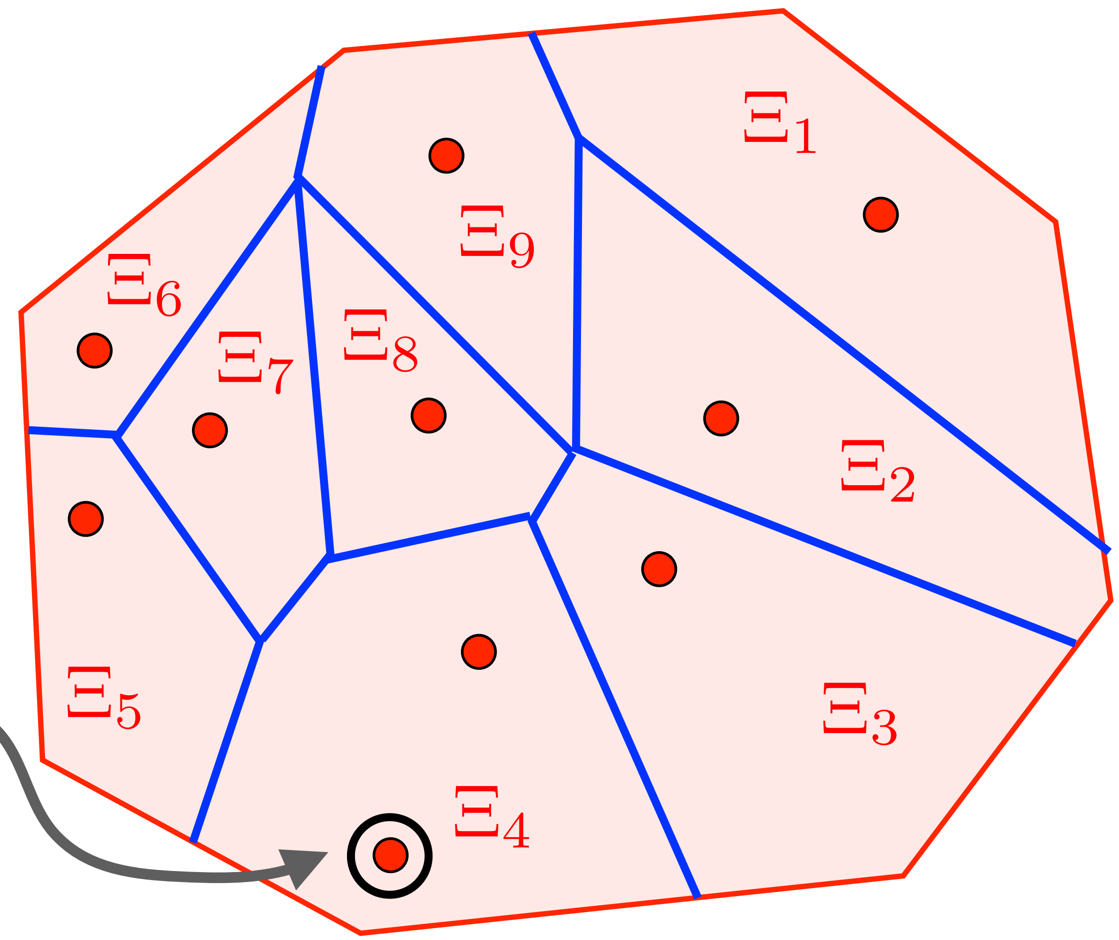
WIKIPEDIA
The Free Encyclopedia

A **Voronoi diagram** is a partition of a plane into regions close to each of a given set of **seeds**. For each seed there is a corresponding region, called a **Voronoi cell**, consisting of all points of the plane closer to that seed than to any other.

Iterative Partitioning: Voronoi Partitioning

We can partition the uncertainty set into a Voronoi diagram:

new active scenario after re-optimization



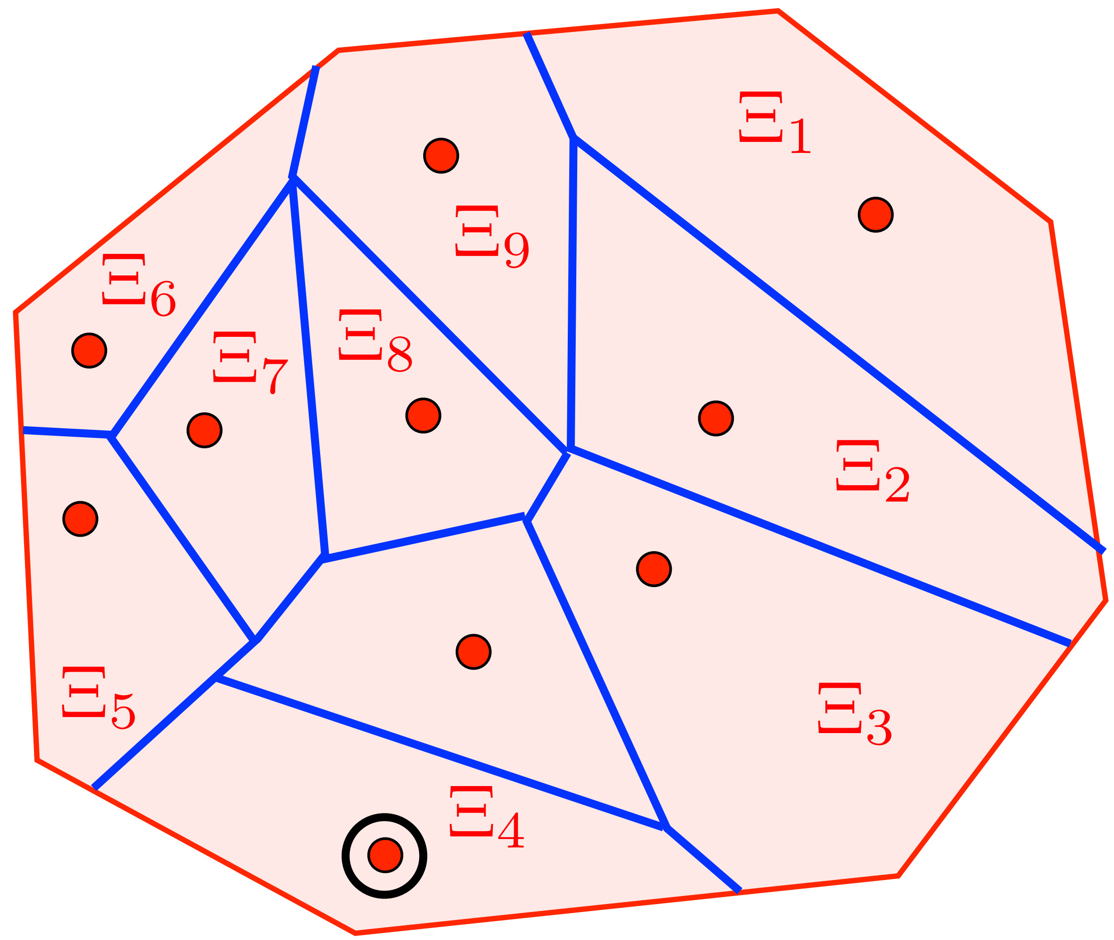
Georgy Feodosevich Voronoy

WIKIPEDIA
The Free Encyclopedia

A **Voronoi diagram** is a partition of a plane into regions close to each of a given set of **seeds**. For each seed there is a corresponding region, called a **Voronoi cell**, consisting of all points of the plane closer to that seed than to any other.

Iterative Partitioning: Voronoi Partitioning

We can partition the uncertainty set into a Voronoi diagram:



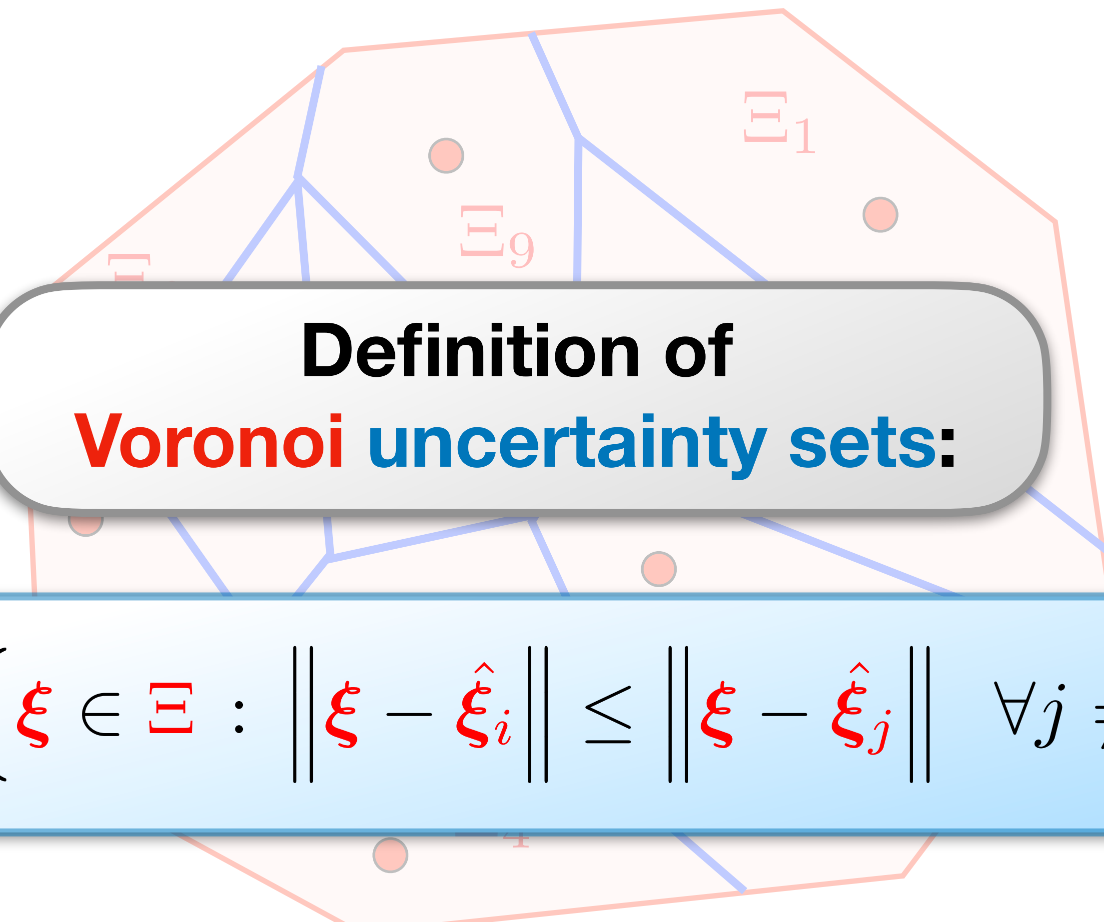
WIKIPEDIA
The Free Encyclopedia

A **Voronoi diagram** is a partition of a plane into regions close to each of a given set of **seeds**. For each seed there is a corresponding region, called a **Voronoi cell**, consisting of all points of the plane closer to that seed than to any other.

Georgy Feodosevich Voronoy

Iterative Partitioning: Voronoi Partitioning

We can partition the uncertainty set into a Voronoi diagram:



Definition of Voronoi uncertainty sets:

$$\Xi_i = \left\{ \boldsymbol{\xi} \in \Xi \, : \, \left\| \boldsymbol{\xi} - \hat{\boldsymbol{\xi}}_i \right\| \leq \left\| \boldsymbol{\xi} - \hat{\boldsymbol{\xi}}_j \right\| \quad \forall j \neq i \right\}$$

Georgy Feodosevich Voronoy

WIKIPEDIA
The Free Encyclopedia

A **Voronoi diagram** is a partition of a plane into regions close to each of a given set of **seeds**. For each seed there is a corresponding region, called a **Voronoi cell**, consisting of all points of the plane closer to that seed than to any other.

Iterative Partitioning: Comparison

Splitting Heuristics	Voronoi Partitioning

Iterative Partitioning: Comparison

	Splitting Heuristics	**Voronoi Partitioning**
Convenience	 ***need to specify splits manually***	 ***splits determined by Voronoi***

Iterative Partitioning: Comparison

	Splitting Heuristics	Voronoi Partitioning
Convenience	2 ***need to specify splits manually***	1 ***splits determined by Voronoi***
Tractability	1 ***new scenarios only have local impact***	2 ***new scenarios have global impact***

Part 2 **Continuous Recourse Decisions**

Two-Stage Models

- Decision Rules
- Lower Bounds
- Benders' Decomposition
- Column-and-Constraint Generation
- Iterative Partitioning
- **Fourier-Motzkin Elimination**

Zhen, den Hertog and Sim. *Adjustable Robust Optimization via Fourier–Motzkin Elimination*. OR, 2018.

Recall the decision rule formulation of the two-stage RO problem:

$$
\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}
\end{aligned}
$$

Jean-Baptiste Joseph Fourier
École Normale Supérieure, École Polytechnique
(1768-1830)

Theodore Motzkin
University of Basel, UCLA
(1908-1970)

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

Pick a second-stage decision $y_k(\boldsymbol{\xi})$ and rewrite the constraints as:

1. $$y_k(\boldsymbol{\xi}) \geq \frac{1}{W_{ik}} \cdot \left[\boldsymbol{h}_i(\boldsymbol{\xi}) - \boldsymbol{T}_i(\boldsymbol{\xi})^\top \boldsymbol{x} - \boldsymbol{W}_i^{-k\top} \boldsymbol{y}_{-k}(\boldsymbol{\xi})\right] \quad \forall \boldsymbol{\xi} \in \Xi,\ \forall i : W_{ik} > 0$$

2. $$y_k(\boldsymbol{\xi}) \leq \frac{1}{|W_{ik}|} \cdot \left[\boldsymbol{T}_i(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{W}_i^{-k\top} \boldsymbol{y}_{-k}(\boldsymbol{\xi}) - \boldsymbol{h}_i(\boldsymbol{\xi})\right] \quad \forall \boldsymbol{\xi} \in \Xi,\ \forall i : W_{ik} < 0$$

3. $$\boldsymbol{T}_i(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{W}_i^{-k\top} \boldsymbol{y}_{-k}(\boldsymbol{\xi}) \geq \boldsymbol{h}_i(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi,\ \forall i : W_{ik} = 0$$

$\boldsymbol{W}_i^{-k\top}$**:** *i*-th row of recourse matrix w/o *k*-th column

Recall the decision rule formulation of the two-stage RO problem:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y} \end{array}$$

Pick a second-stage decision $y_k(\boldsymbol{\xi})$ and rewrite the constraints as:

1. $$y_k(\boldsymbol{\xi}) \geq \frac{1}{W_{ik}} \cdot \left[\boldsymbol{h}_i(\boldsymbol{\xi}) - \boldsymbol{T}_i(\boldsymbol{\xi})^\top \boldsymbol{x} - \boldsymbol{W}_i^{-k\top} \boldsymbol{y}_{-k}(\boldsymbol{\xi})\right] \quad \forall \boldsymbol{\xi} \in \Xi,\ \forall i : W_{ik} > 0$$
2. $$y_k(\boldsymbol{\xi}) \leq \frac{1}{|W_{ik}|} \cdot \left[\boldsymbol{T}_i(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{W}_i^{-k\top} \boldsymbol{y}_{-k}(\boldsymbol{\xi}) - \boldsymbol{h}_i(\boldsymbol{\xi})\right] \quad \forall \boldsymbol{\xi} \in \Xi,\ \forall i : W_{ik} < 0$$
3. $$\boldsymbol{T}_i(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{W}_i^{-k\top} \boldsymbol{y}_{-k}(\boldsymbol{\xi}) \geq \boldsymbol{h}_i(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi,\ \forall i : W_{ik} = 0$$

Eliminate $y_k(\boldsymbol{\xi})$ by bounding all combinations of 1 and 2:

$$\left.\begin{array}{l} \dfrac{1}{W_{ik}} \cdot \left[\boldsymbol{h}_i(\boldsymbol{\xi}) - \boldsymbol{T}_i(\boldsymbol{\xi})^\top \boldsymbol{x} - \boldsymbol{W}_i^{-k\top} \boldsymbol{y}_{-k}(\boldsymbol{\xi})\right] \\ \qquad \leq \dfrac{1}{|W_{jk}|} \cdot \left[\boldsymbol{T}_j(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{W}_j^{-k\top} \boldsymbol{y}_{-k}(\boldsymbol{\xi}) - \boldsymbol{h}_j(\boldsymbol{\xi})\right] \end{array}\right\} \begin{array}{c} \forall \boldsymbol{\xi} \in \Xi,\ \forall i : W_{ik} > 0, \\ \forall j : W_{jk} < 0 \end{array}$$

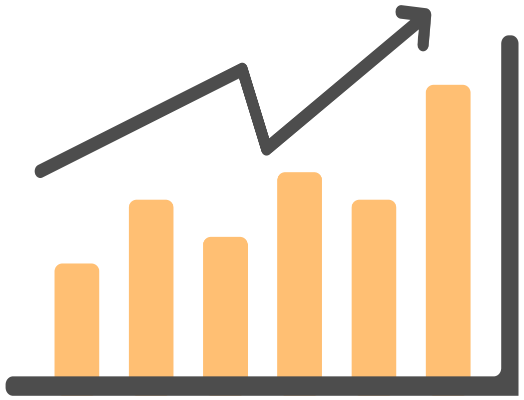
Fourier-Motzkin Elimination

 method can be combined with other approximations, such as linear decision rules

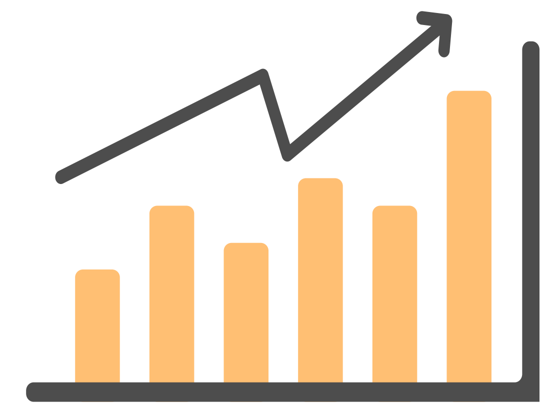
Fourier-Motzkin Elimination

- method can be combined with other approximations, such as linear decision rules

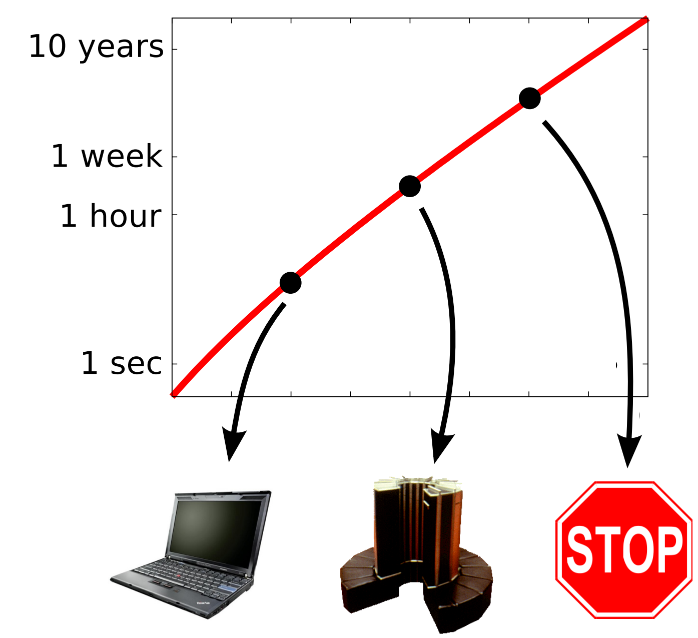
- various practical improvements available, such as dropping of redundant constraints



- method can be combined with other approximations, such as linear decision rules



- various practical improvements available, such as dropping of redundant constraints



- worst-case complexity is high: $\mathcal{O}(M^{2^N})$ if we eliminate N second-stage decision variables from M constraints

Part 3

Continuous Recourse Decisions

Multi-Stage Models

- **Time (In-)Consistency**
- Decision Rules
- Iterative Partitioning
- Nested Benders’ Decomposition
- Robust Dual Dynamic Programming

Delage and Iancu. *Robust Multistage Decision Making*. INFORMS Tutorials in OR, 2015.

Time Consistency (Informal Definition)

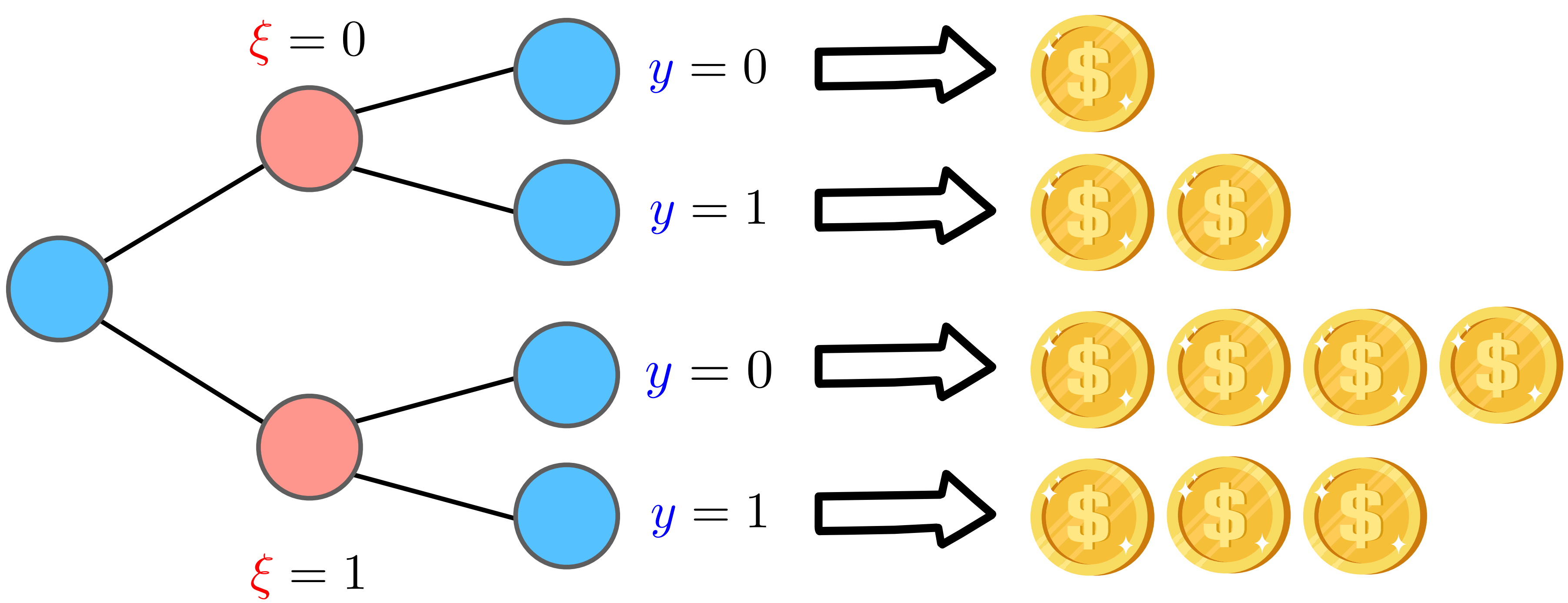
At no point in future does the decision maker prefer to deviate from any of the decisions suggested by the robust optimization model that is solved today.

Time Consistency (Informal Definition)

At no point in future does the decision maker prefer to deviate from any of the decisions suggested by the robust optimization model that is solved today.

RO problems suffer from two types of time *in*consistency:

1. Suboptimal decisions in non-worst-case scenarios $\xi \in \Xi$:

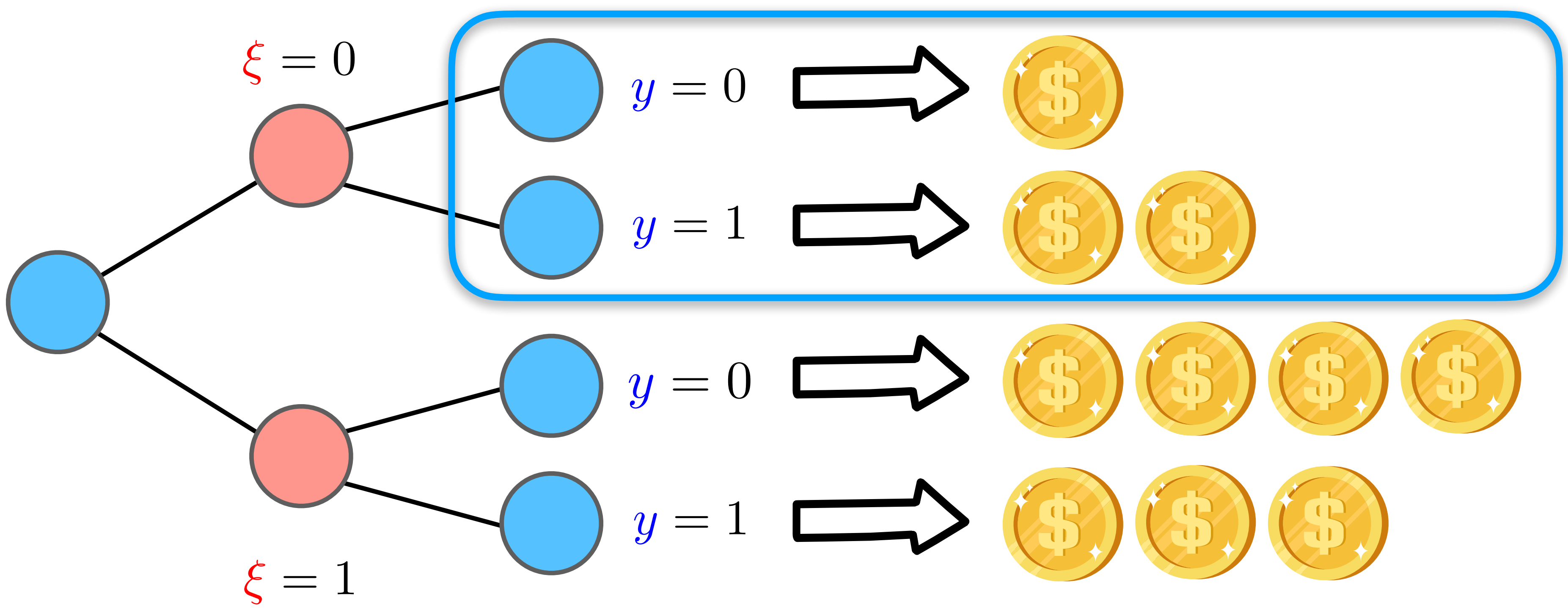


Time Consistency (Informal Definition)

At no point in future does the decision maker prefer to deviate from any of the decisions suggested by the robust optimization model that is solved today.

RO problems suffer from two types of time *in*consistency:

1. Suboptimal decisions in non-worst-case scenarios $\xi \in \Xi$:

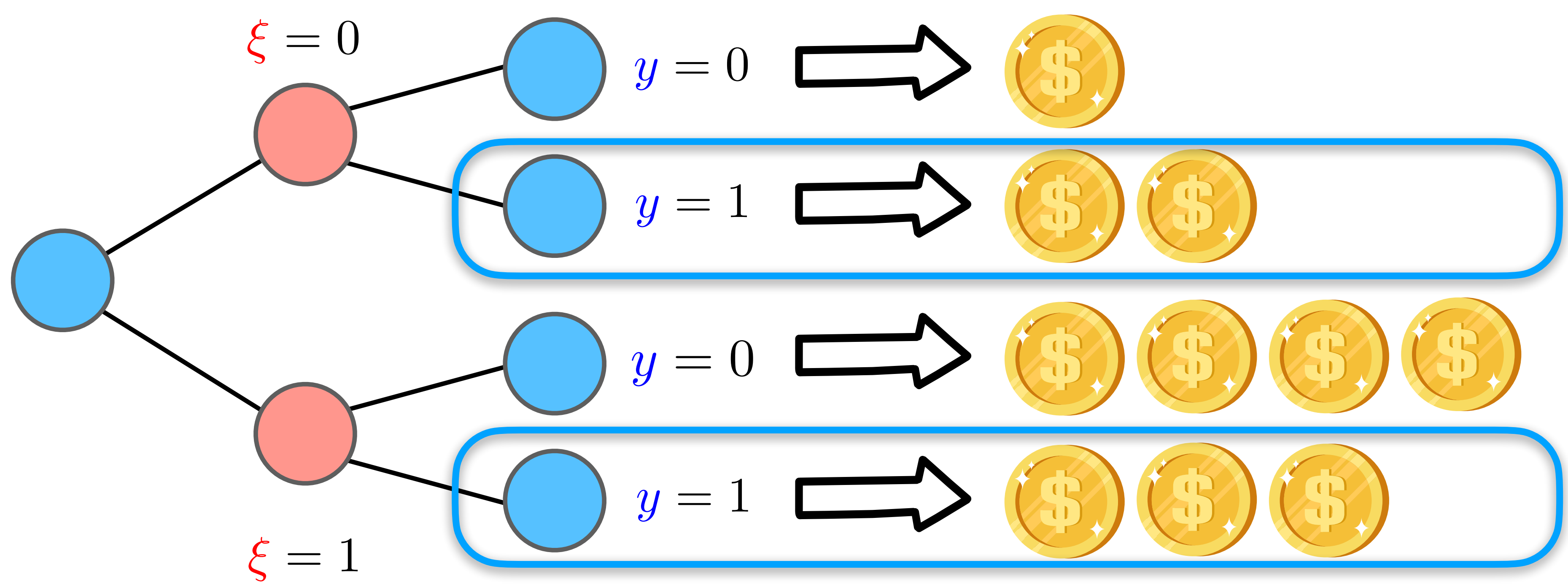


Time Consistency (Informal Definition)

At no point in future does the decision maker prefer to deviate from any of the decisions suggested by the robust optimization model that is solved today.

RO problems suffer from two types of time *in*consistency:

1. Suboptimal decisions in non-worst-case scenarios $\xi \in \Xi$:



(a) worst-case optimal strategy: always choose $y = 1$

Time Consistency (Informal Definition)

At no point in future does the decision maker prefer to deviate from any of the decisions suggested by the robust optimization model that is solved today.

RO problems suffer from two types of time *in*consistency:

1. Suboptimal decisions in non-worst-case scenarios $\xi \in \Xi$:

Fix: resolve RO problem in every stage
(Optimal value is guaranteed to weakly improve.)

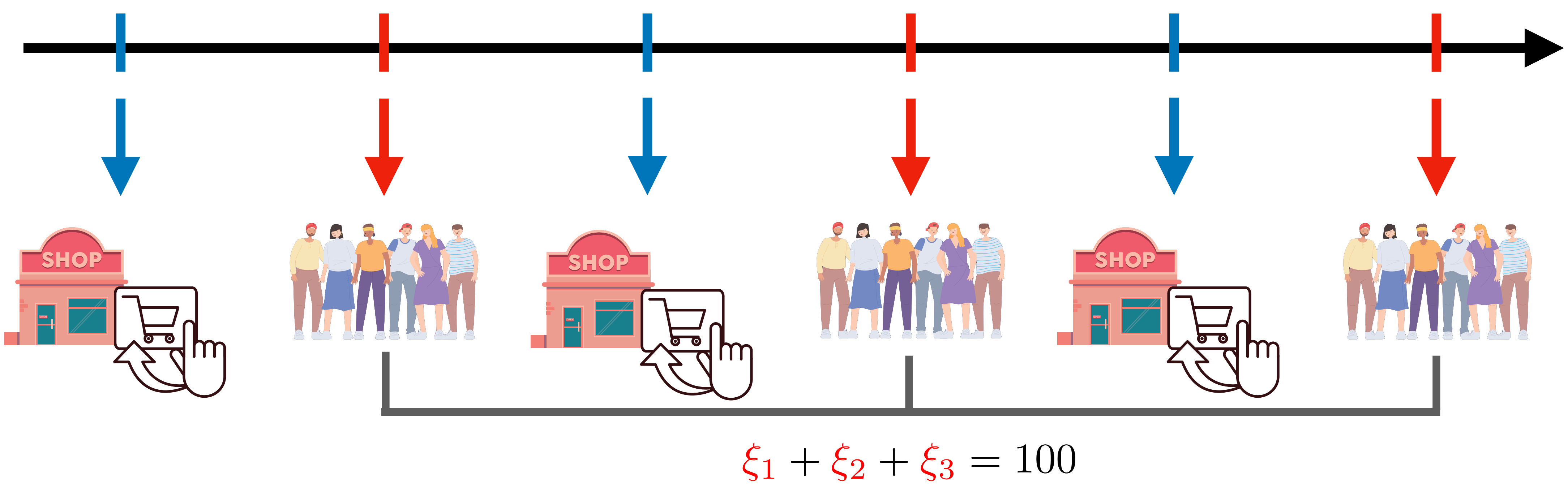
(a) worst-case optimal strategy: always choose $y = 1$

Time Consistency (Informal Definition)

At no point in future does the decision maker prefer to deviate from any of the decisions suggested by the robust optimization model that is solved today.

RO problems suffer from two types of time *in*consistency:

1. Suboptimal decisions in non-worst-case scenarios $\xi \in \Xi$.
2. Uncertainty set Ξ is updated as time progresses:

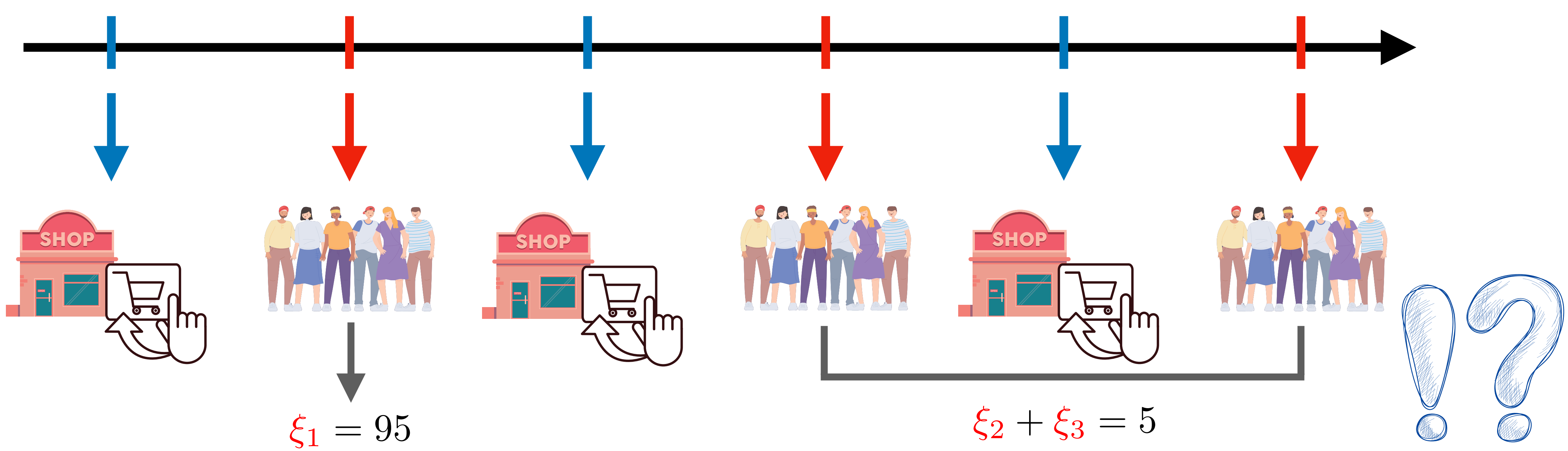


Time Consistency (Informal Definition)

At no point in future does the decision maker prefer to deviate from any of the decisions suggested by the robust optimization model that is solved today.

RO problems suffer from two types of time *in*consistency:

1. Suboptimal decisions in non-worst-case scenarios $\xi \in \Xi$.
2. Uncertainty set Ξ is updated as time progresses:



Time Consistency (Informal Definition)

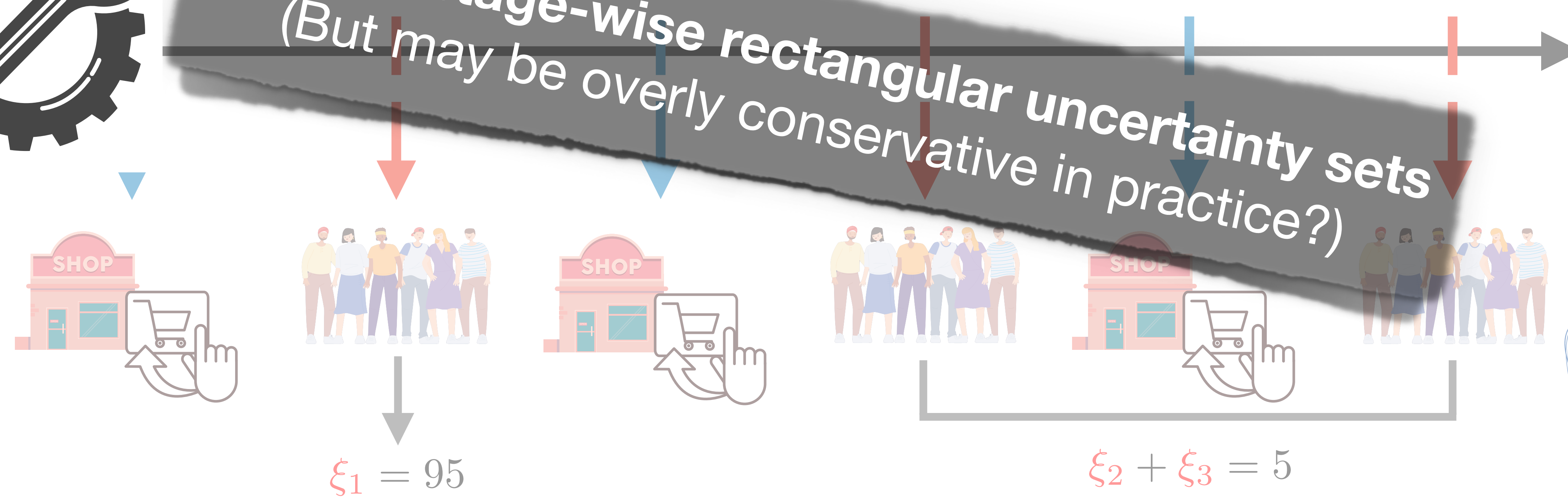
At no point in future does the decision maker prefer to deviate from any of the decisions suggested by the robust optimization model that is solved today.

RO problems suffer from two types of time *in*consistency:

1. Suboptimal decisions in non-worst-case scenarios $\xi \in \Xi$.

2. Uncertainty set Ξ is updated as time progresses:

Fix: use stage-wise rectangular uncertainty sets
(But may be overly conservative in practice?)



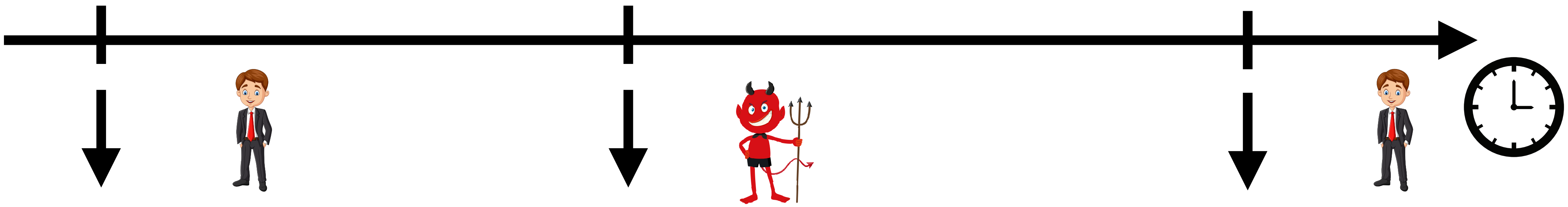
Part 3 **Continuous Recourse Decisions**

Multi-Stage Models

- Time (In-)Consistency
- **Decision Rules**
- Iterative Partitioning
- Nested Benders' Decomposition
- Robust Dual Dynamic Programming

Delage and Iancu. *Robust Multistage Decision Making*. INFORMS Tutorials in OR, 2015.
Yanıkoğlu, Gorissen and den Hertog. *Adjustable Robust Optimization–A Survey and Tutorial*. EJOR, 2019.

Recall the two-stage robust optimization problem:

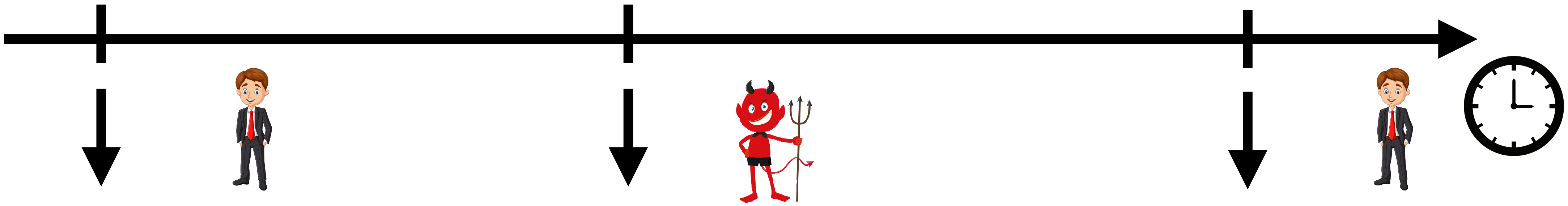


$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Decision Rules

Recall the two-stage robust optimization problem:



$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \; + \; \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \; + \; \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boxed{\boldsymbol{y} \in \mathcal{Y}} \end{array}$$

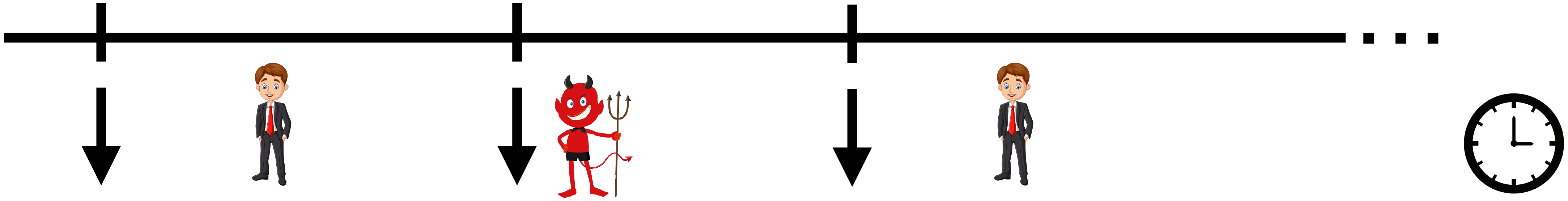
$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

Move second-stage decisions to first stage via decision rules:

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \theta & \\ \text{subject to} & \theta \geq \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} + \boldsymbol{r}^\top \boldsymbol{y}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \; \boxed{\boldsymbol{y} : \Xi \mapsto \mathcal{Y}} & \end{array}$$

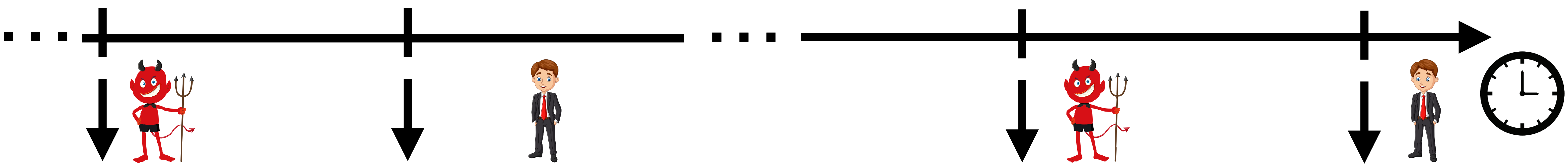
Decision Rules

We can do the same in multi-stage robust optimization:



$$\begin{array}{ll} \min & \boldsymbol{q}_1^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1 \end{array} + \max_{\boldsymbol{\xi}_2 \in \Xi_2} \begin{array}{ll} \min & \boldsymbol{q}_2^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{x}_2 \in \mathcal{X}_2 \end{array} +$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}^2)\, \boldsymbol{x}_1 + \boldsymbol{W}_2 \boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}^2)$$



$$\max_{\boldsymbol{\xi}_3 \in \Xi_3(\boldsymbol{\xi}^2)} \begin{array}{ll} \min & \boldsymbol{q}_3^\top \boldsymbol{x}_3 \\ \text{s.t.} & \boldsymbol{x}_3 \in \mathcal{X}_3 \end{array} + \cdots + \max_{\boldsymbol{\xi}_T \in \Xi_T(\boldsymbol{\xi}^{T-1})} \begin{array}{ll} \min & \boldsymbol{q}_T^\top \boldsymbol{x}_T \\ \text{s.t.} & \boldsymbol{x}_T \in \mathcal{X}_T \end{array}$$

$$\boldsymbol{T}_3(\boldsymbol{\xi}^3)\, \boldsymbol{x}_2 + \boldsymbol{W}_3 \boldsymbol{x}_3 \geq \boldsymbol{h}_3(\boldsymbol{\xi}^3)$$

$$\boldsymbol{T}_T(\boldsymbol{\xi}^T)\, \boldsymbol{x}_{T-1} + \boldsymbol{W}_T \boldsymbol{x}_T \geq \boldsymbol{h}_T(\boldsymbol{\xi}^T)$$

We can do the same in multi-stage robust optimization:

$$\begin{array}{ll} \min & \boldsymbol{q}_1^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1 \end{array} + \max_{\boldsymbol{\xi}_2 \in \Xi_2} \begin{array}{ll} \min & \boldsymbol{q}_2^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boxed{\boldsymbol{x}_2 \in \mathcal{X}_2} \end{array} +$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}^2)\,\boldsymbol{x}_1 + \boldsymbol{W}_2 \boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}^2)$$

$$\max_{\boldsymbol{\xi}_3 \in \Xi_3(\boldsymbol{\xi}^2)} \begin{array}{ll} \min & \boldsymbol{q}_3^\top \boldsymbol{x}_3 \\ \text{s.t.} & \boxed{\boldsymbol{x}_3 \in \mathcal{X}_3} \end{array} + \cdots + \max_{\boldsymbol{\xi}_T \in \Xi_T(\boldsymbol{\xi}^{T-1})} \begin{array}{ll} \min & \boldsymbol{q}_T^\top \boldsymbol{x}_T \\ \text{s.t.} & \boxed{\boldsymbol{x}_T \in \mathcal{X}_T} \end{array}$$

$$\boldsymbol{T}_3(\boldsymbol{\xi}^3)\,\boldsymbol{x}_2 + \boldsymbol{W}_3 \boldsymbol{x}_3 \geq \boldsymbol{h}_3(\boldsymbol{\xi}^3)$$

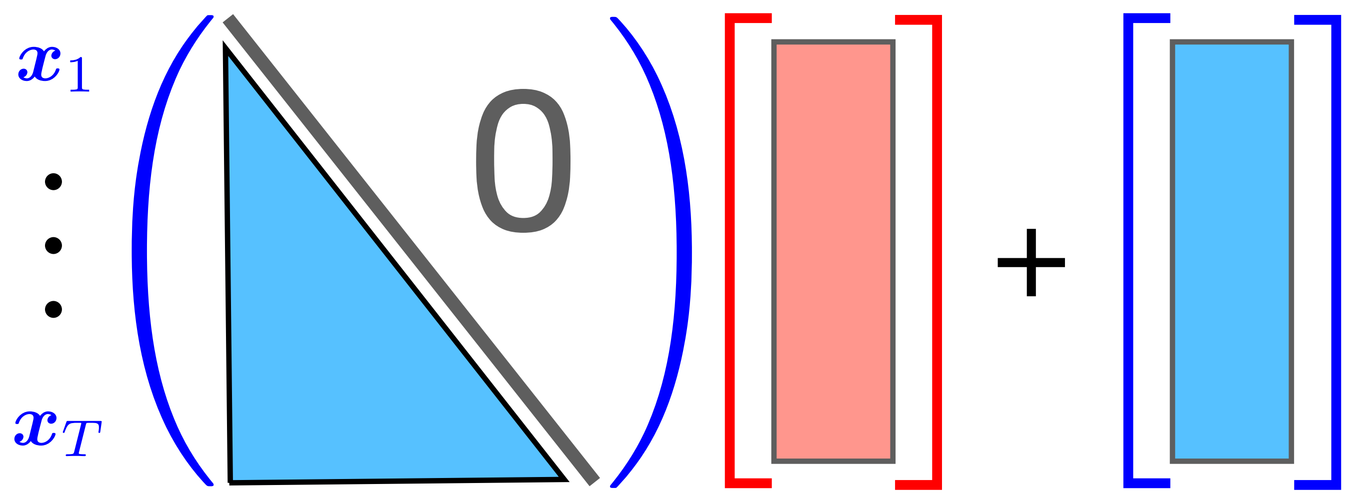
$$\boldsymbol{T}_T(\boldsymbol{\xi}^T)\,\boldsymbol{x}_{T-1} + \boldsymbol{W}_T \boldsymbol{x}_T \geq \boldsymbol{h}_T(\boldsymbol{\xi}^T)$$

We can do the same in multi-stage robust optimization:

$$
\begin{array}{lll}
\underset{\{\boldsymbol{x}_t\}_t,\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x}_1 + \theta & \\
\text{subject to} & \theta \geq \displaystyle\sum_{t=2}^{T} \boldsymbol{q}_t^\top \boldsymbol{x}_t(\boldsymbol{\xi}^t) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}_t(\boldsymbol{\xi}^t)\, \boldsymbol{x}_{t-1}(\boldsymbol{\xi}^{t-1}) + \boldsymbol{W}_t \boldsymbol{x}_t(\boldsymbol{\xi}^t) \geq \boldsymbol{h}_t(\boldsymbol{\xi}^t) & \forall \boldsymbol{\xi} \in \Xi, \\
& & \forall t = 2, \ldots, T \\
& \boldsymbol{x}_1 \in \mathcal{X}_1, \;\; \boldsymbol{x}_t : \Xi^t \mapsto \mathcal{X}_t, \; t = 2, \ldots, t &
\end{array}
$$

We can do the same in multi-stage robust optimization:

$$
\begin{array}{lll}
\underset{\{\boldsymbol{x}_t\}_t,\theta}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x}_1 + \theta & \\
\text{subject to} & \theta \geq \displaystyle\sum_{t=2}^{T} \boldsymbol{q}_t^\top \boldsymbol{x}_t(\boldsymbol{\xi}^t) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{T}_t(\boldsymbol{\xi}^t)\, \boldsymbol{x}_{t-1}(\boldsymbol{\xi}^{t-1}) + \boldsymbol{W}_t \boldsymbol{x}_t(\boldsymbol{\xi}^t) \geq \boldsymbol{h}_t(\boldsymbol{\xi}^t) & \forall \boldsymbol{\xi} \in \Xi, \\
& & \forall t = 2, \ldots, T \\
& \boldsymbol{x}_1 \in \mathcal{X}_1, \;\; \boldsymbol{x}_t : \Xi \mapsto \mathcal{X}_t,\; t = 2, \ldots, t &
\end{array}
$$



$\boldsymbol{x} : \Xi \mapsto \mathcal{X}_1 \times \ldots \times \mathcal{X}_T$

Part 3

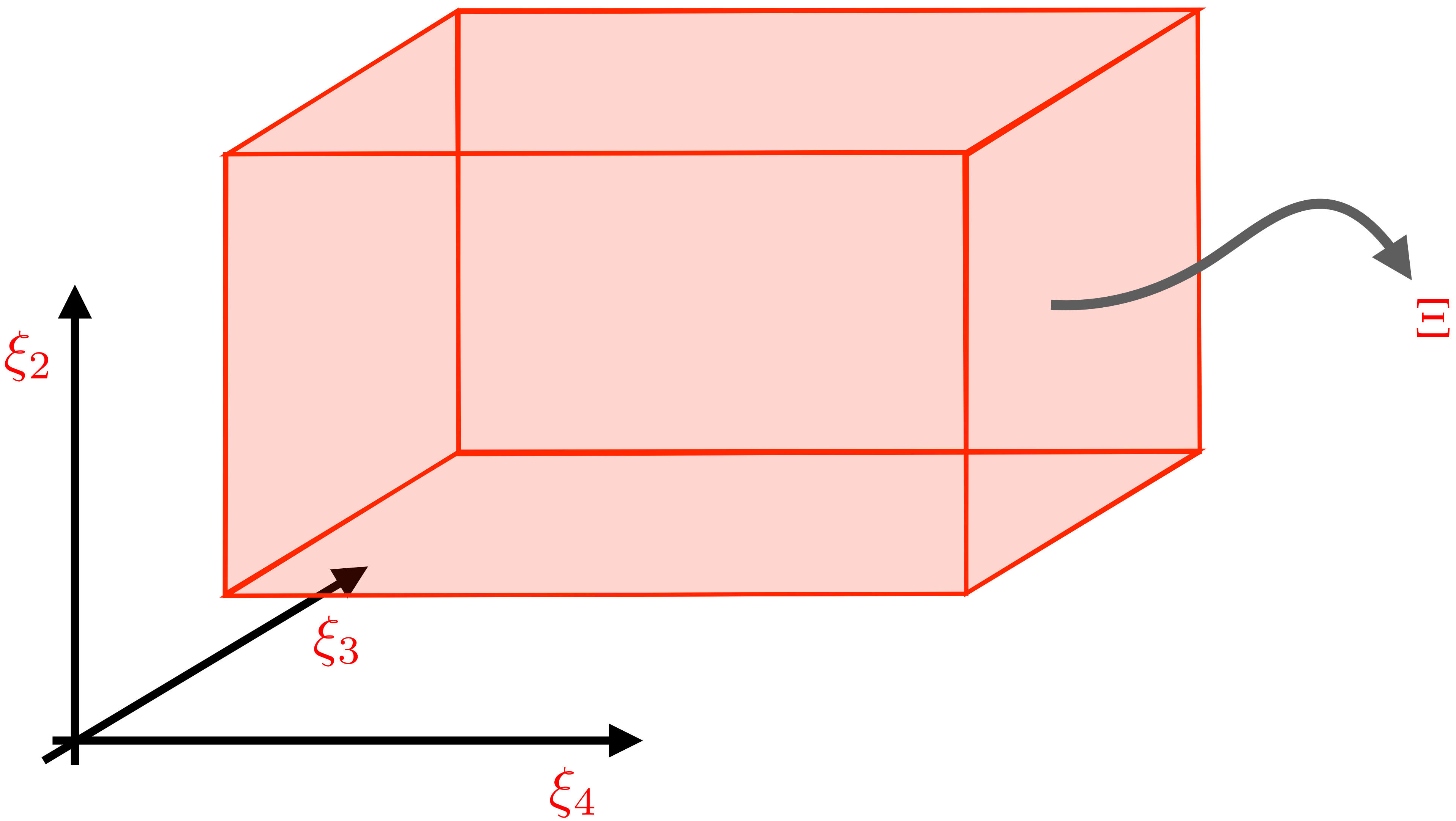
Continuous Recourse Decisions

Multi-Stage Models

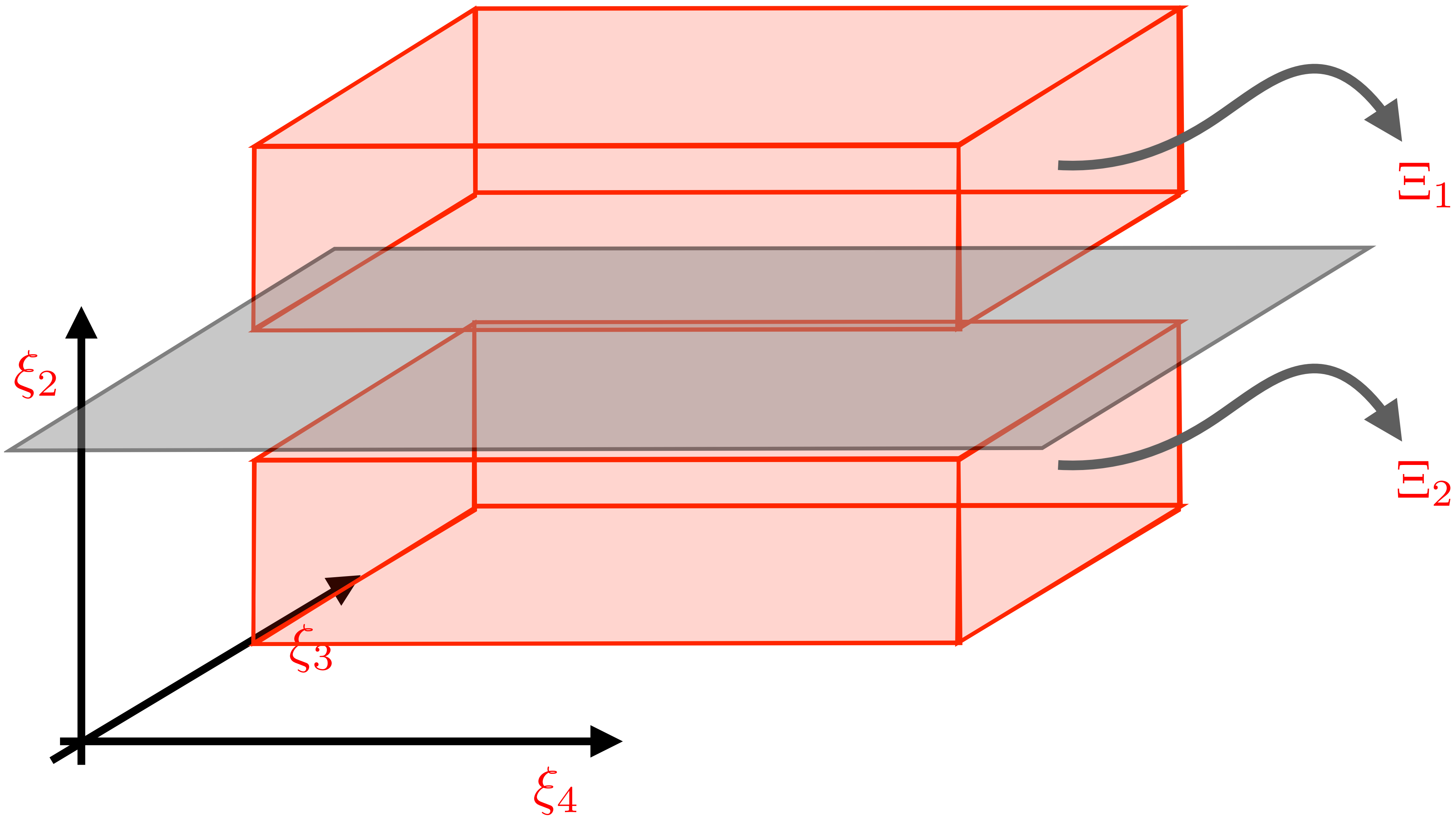
- Time (In-)Consistency
- Decision Rules
- **Iterative Partitioning**
- Nested Benders' Decomposition
- Robust Dual Dynamic Programming

Bertsimas and Dunning. *Multistage Robust Mixed-Integer Optimization with Adaptive Partitions*. OR, 2016.
Postek and den Hertog. *Multistage Adjustable Robust Mixed-Integer Optimization via Iterative Splitting of the Uncertainty Set*. INFORMS JoC, 2016.

Different splits imply different non-anticipativity constraints:

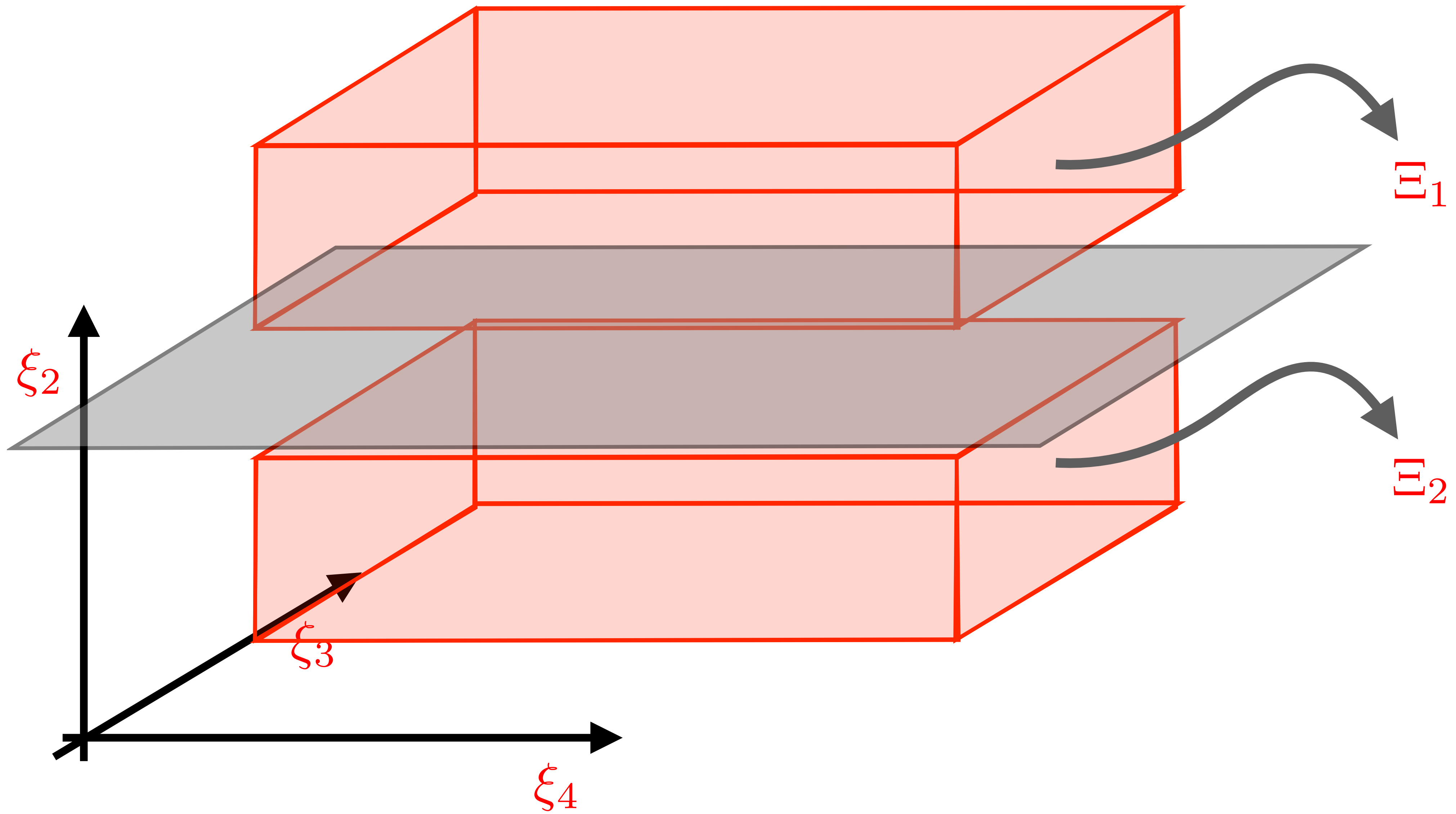


Different splits imply different non-anticipativity constraints:



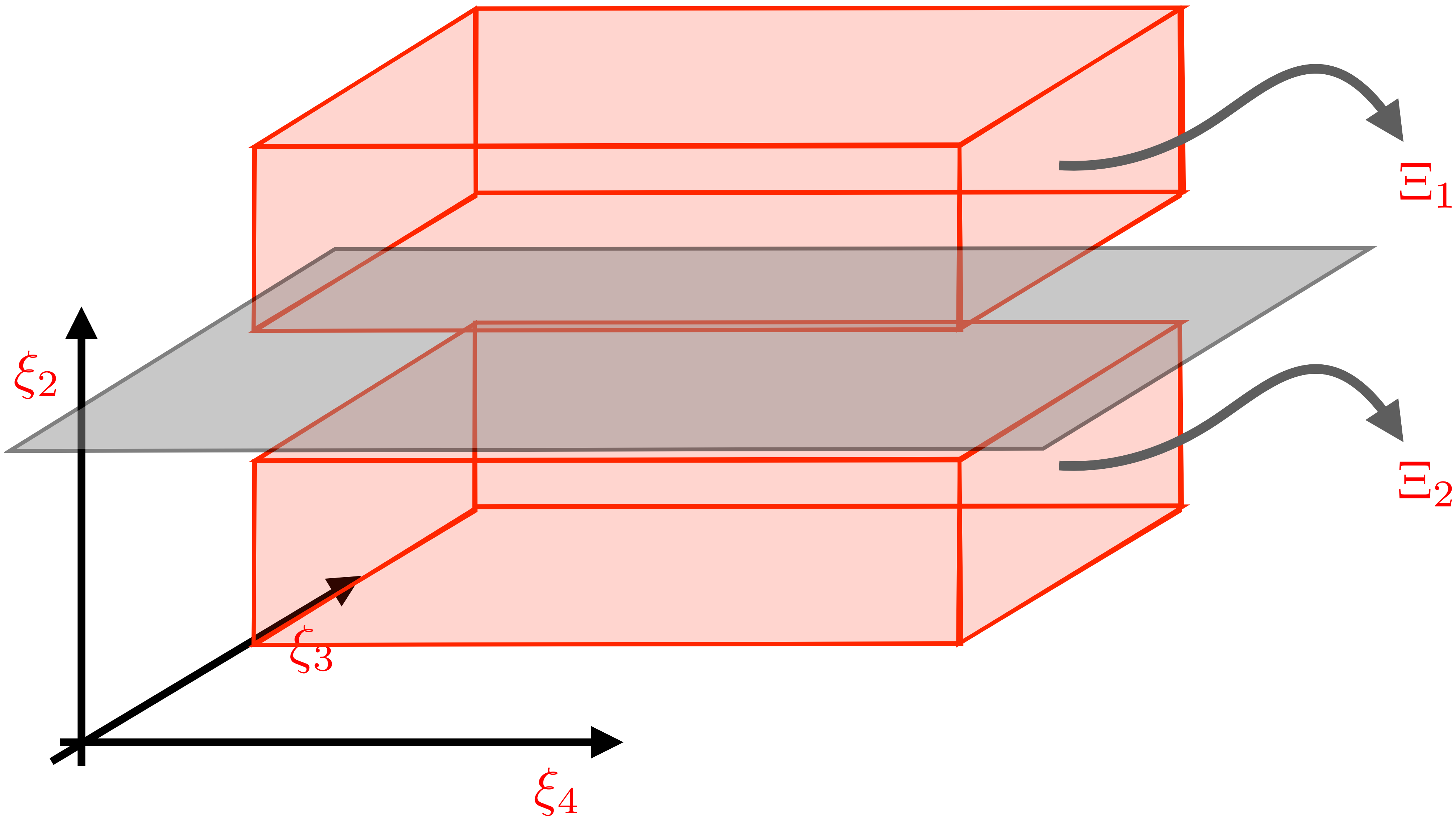
Iterative Partitioning: Splitting Heuristics

Different splits imply different non-anticipativity constraints:



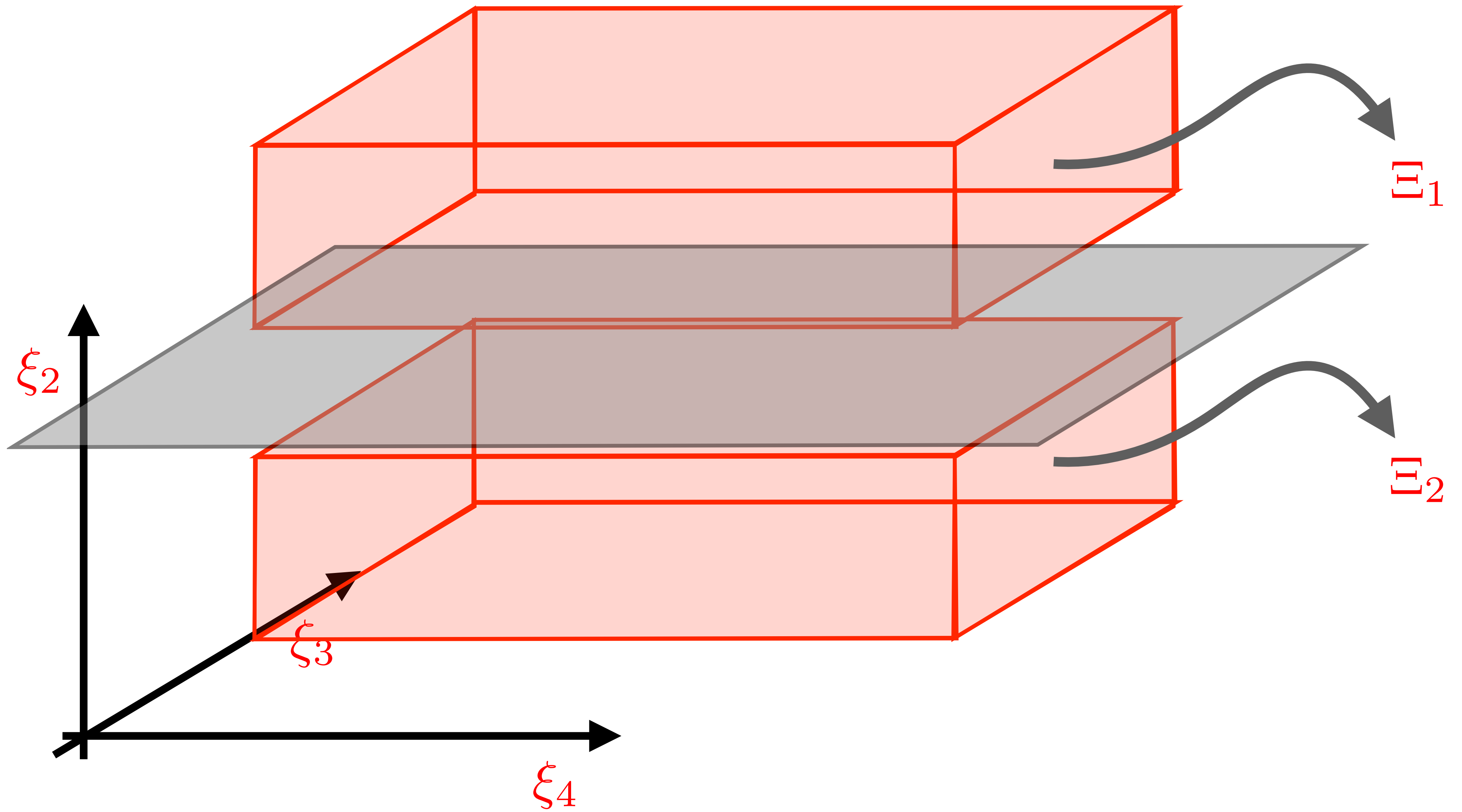
We can distinguish between Ξ_1 and Ξ_2 upon observing ξ_2

Different splits imply different non-anticipativity constraints:



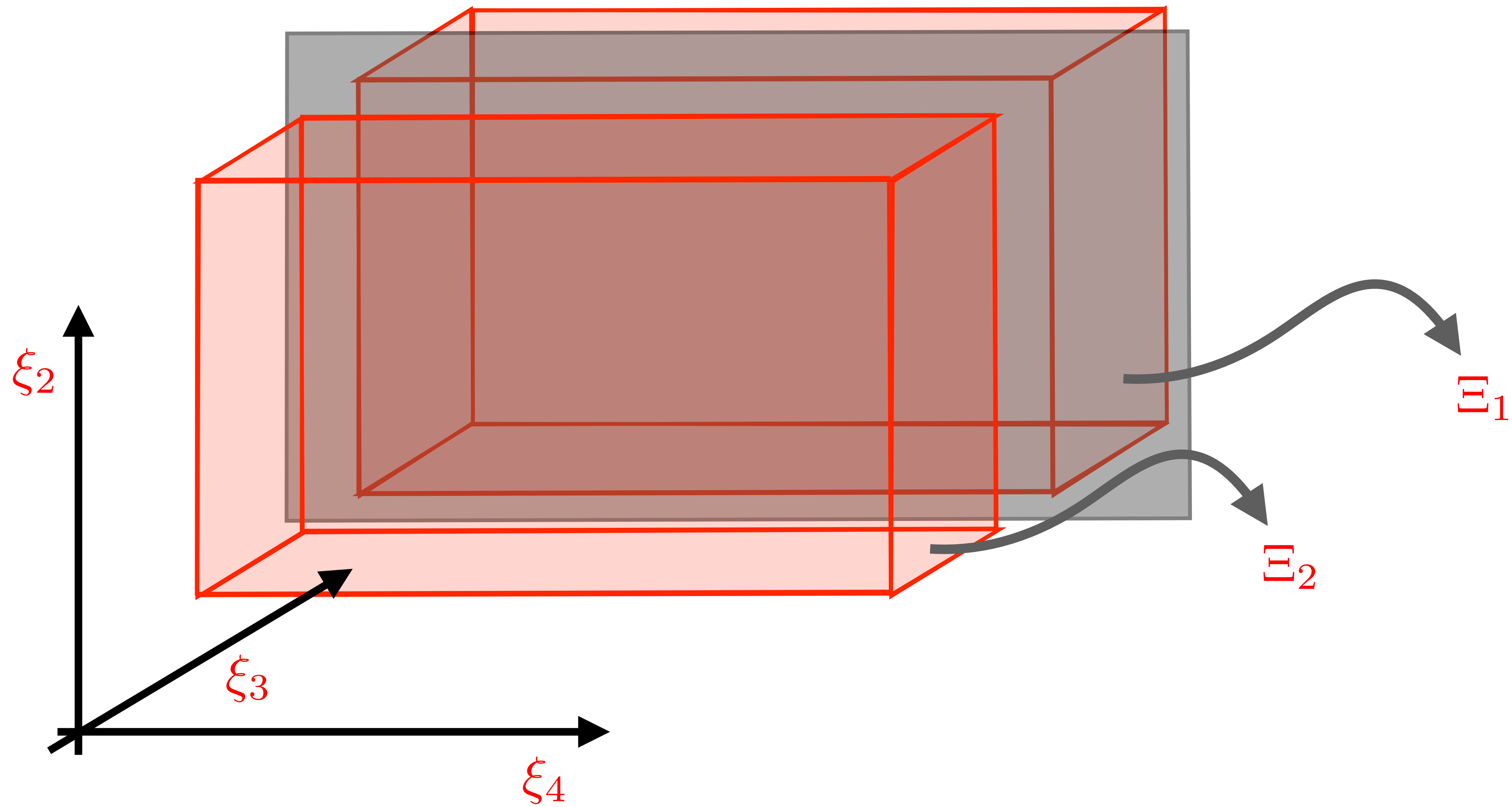
- We can distinguish between Ξ_1 and Ξ_2 upon observing ξ_2
- We have observed ξ_2 by time period $t = 2$

Different splits imply different non-anticipativity constraints:



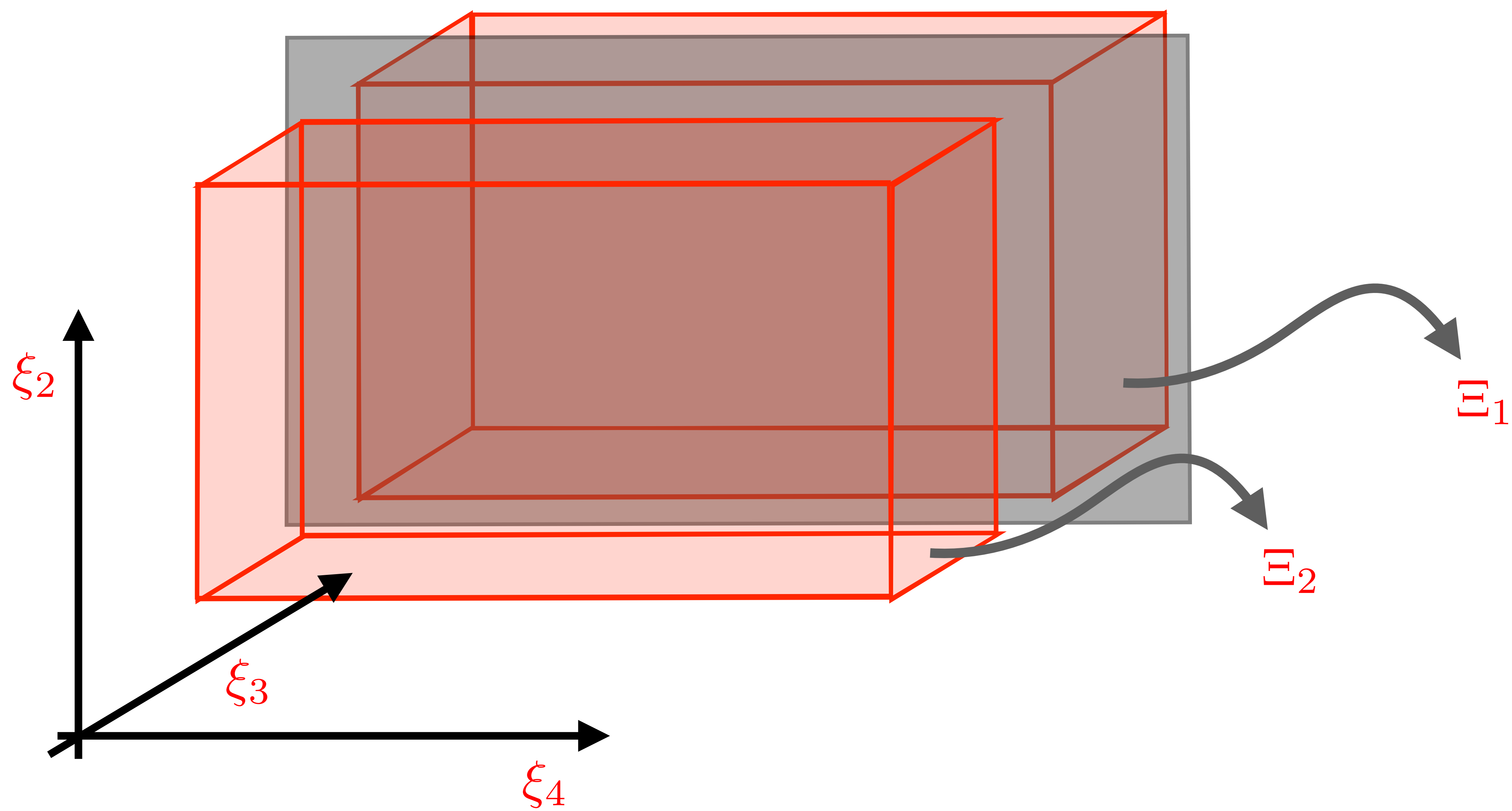
- We can distinguish between Ξ_1 and Ξ_2 upon observing ξ_2
- We have observed ξ_2 by time period $t = 2$
- The first-stage decisions need to be the same across Ξ_1 and Ξ_2

Different splits imply different non-anticipativity constraints:



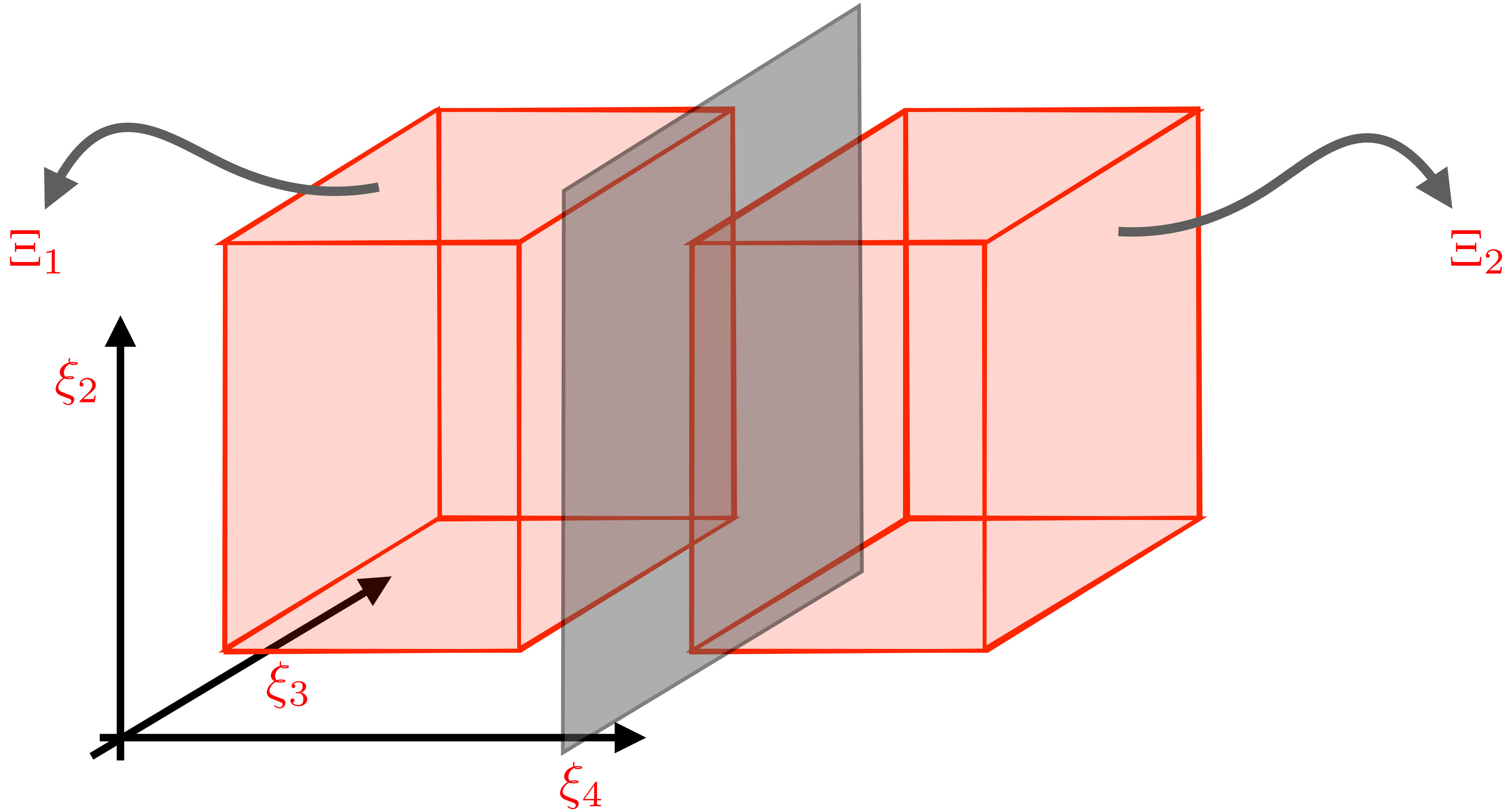
Iterative Partitioning: Splitting Heuristics

Different splits imply different non-anticipativity constraints:



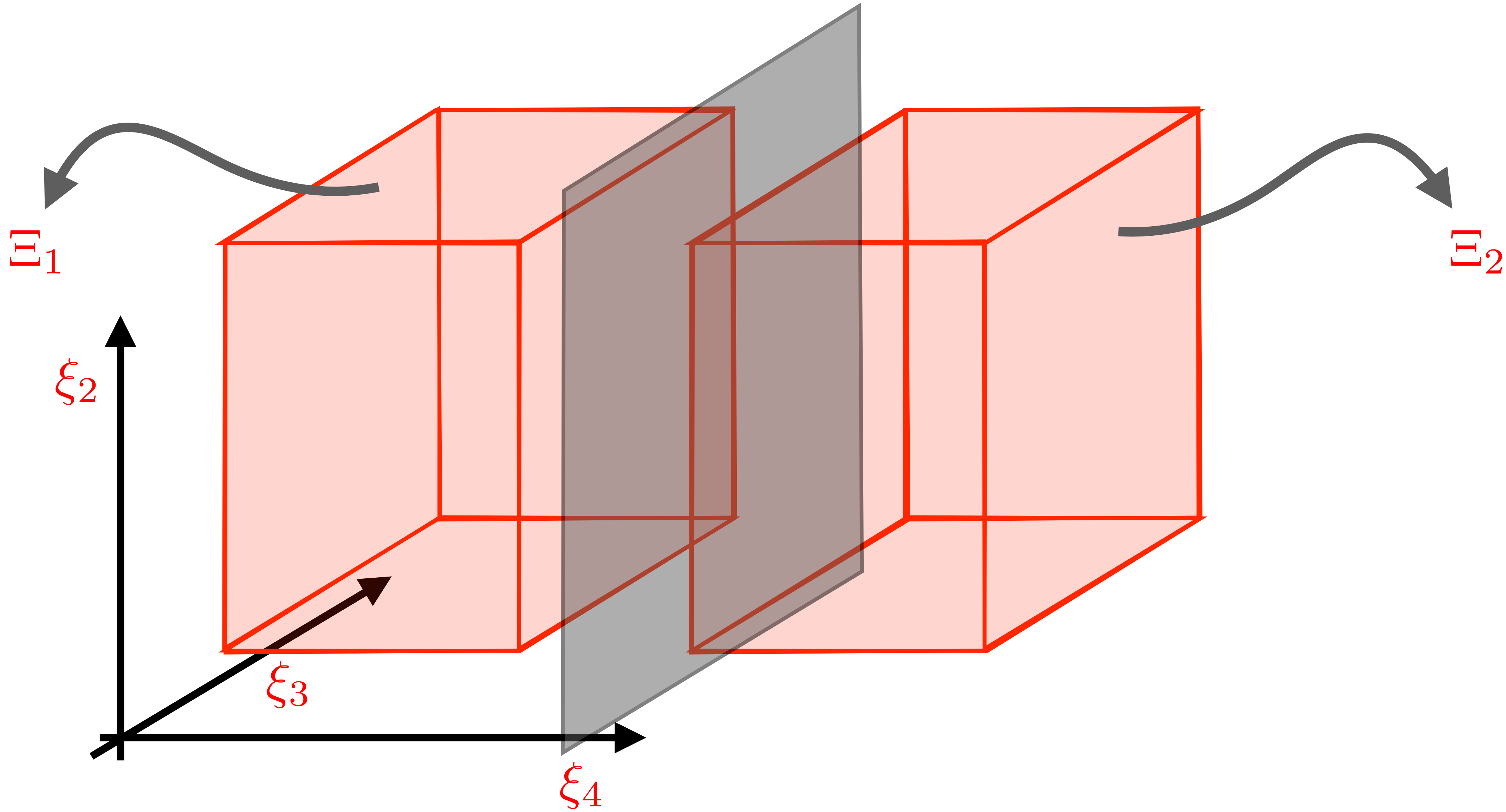
- We can distinguish between Ξ_1 and Ξ_2 upon observing ξ_3
- We have observed ξ_3 by time period $t = 3$
- The stage-1+2 decisions need to be the same across Ξ_1 and Ξ_2

Different splits imply different non-anticipativity constraints:



Iterative Partitioning: Splitting Heuristics

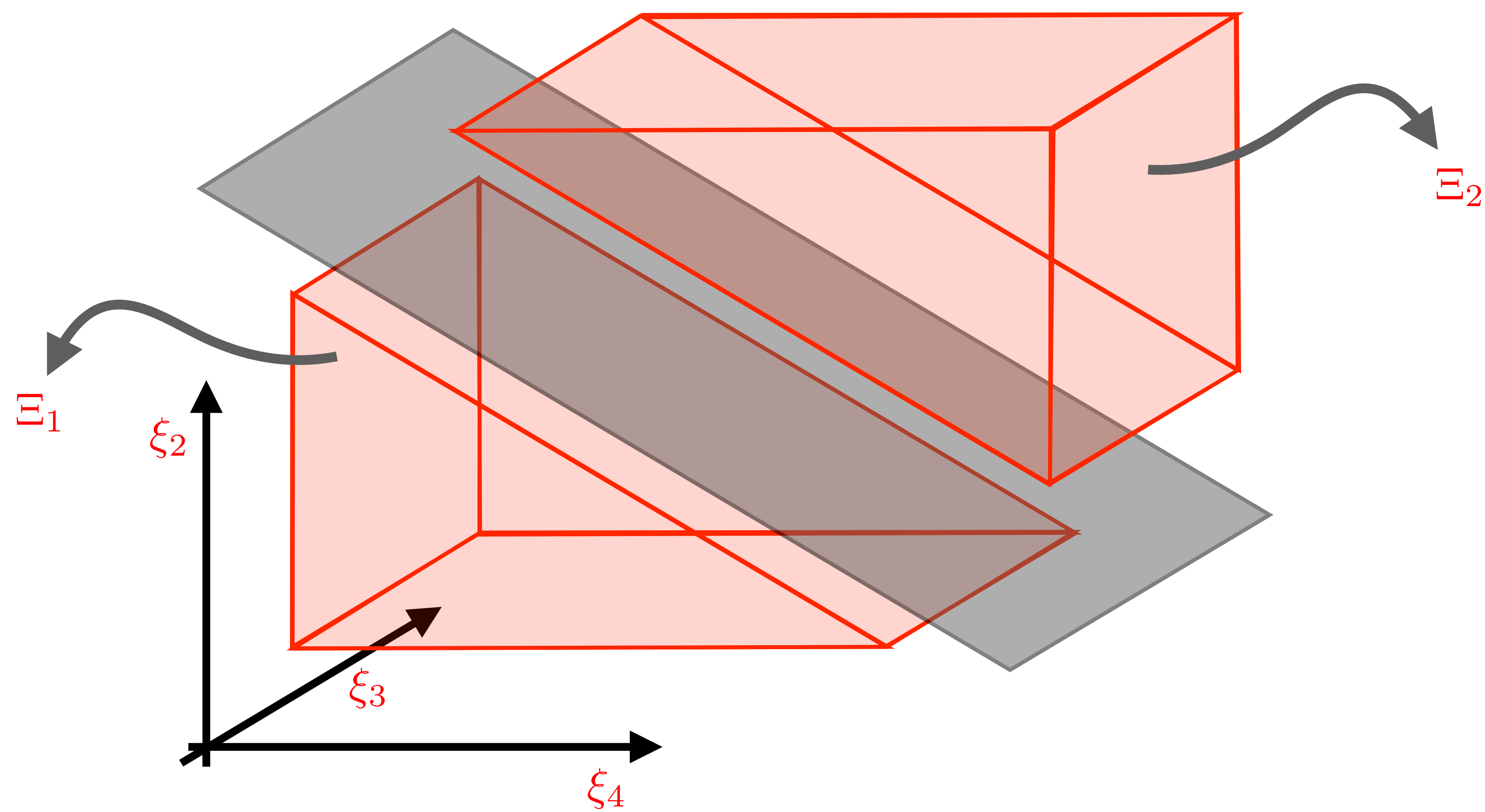
Different splits imply different non-anticipativity constraints:



- We can distinguish between Ξ_1 and Ξ_2 upon observing ξ_4
- We have observed ξ_4 by time period $t = 4$
- The stage-1/2/3 decisions need to be the same across Ξ_1 and Ξ_2

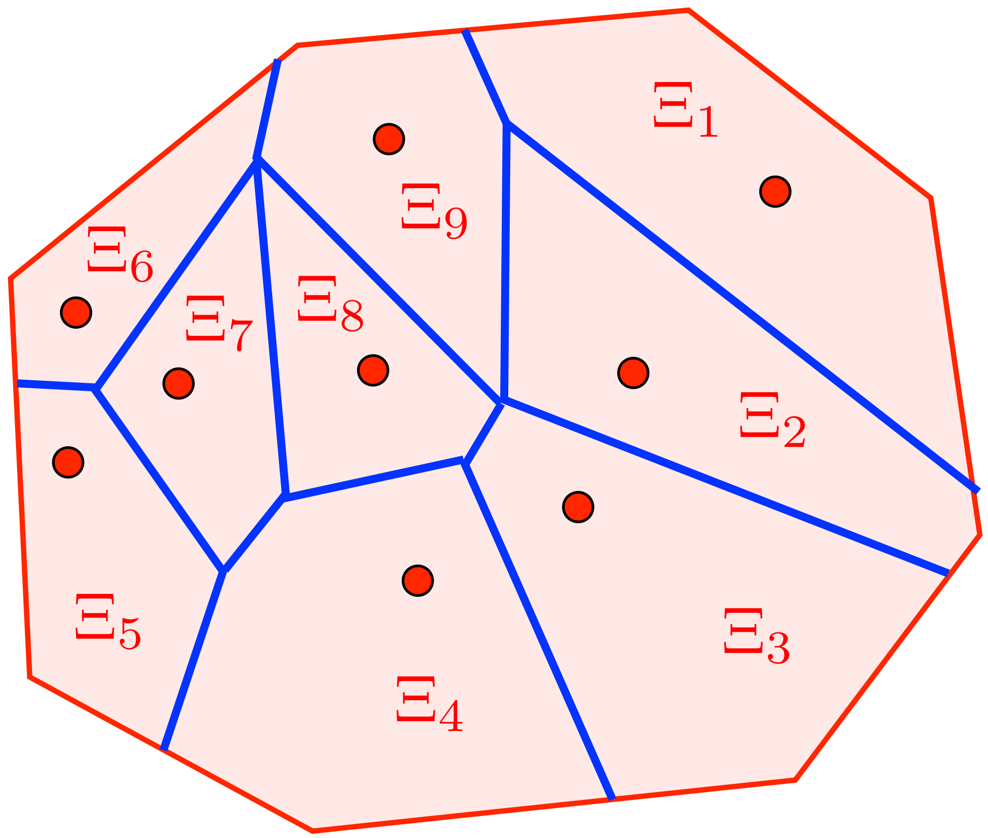
Iterative Partitioning: Splitting Heuristics

Different splits imply different non-anticipativity constraints:



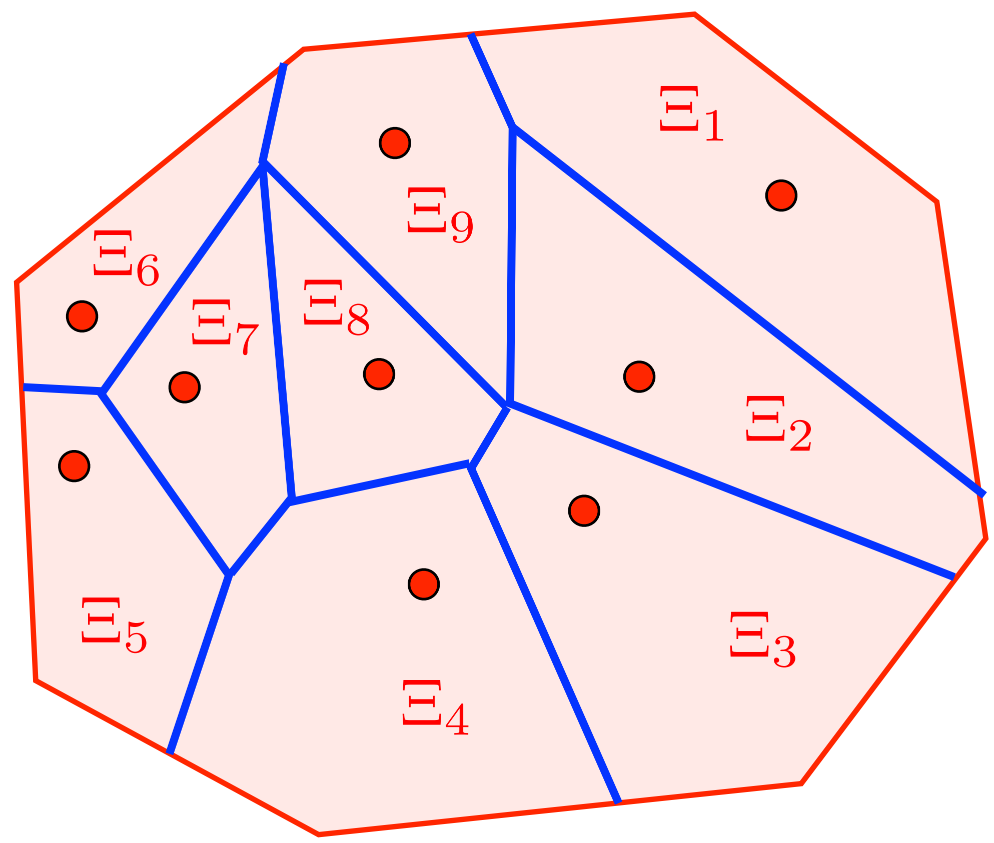
- We can distinguish between Ξ_1 and Ξ_2 upon observing ξ_4
- We have observed ξ_4 by time period $t = 4$
- The stage-1/2/3 decisions need to be the same across Ξ_1 and Ξ_2

Recall the Voronoi partitioning of uncertainty sets:



Iterative Partitioning: Voronoi Partitioning

Recall the Voronoi partitioning of uncertainty sets:



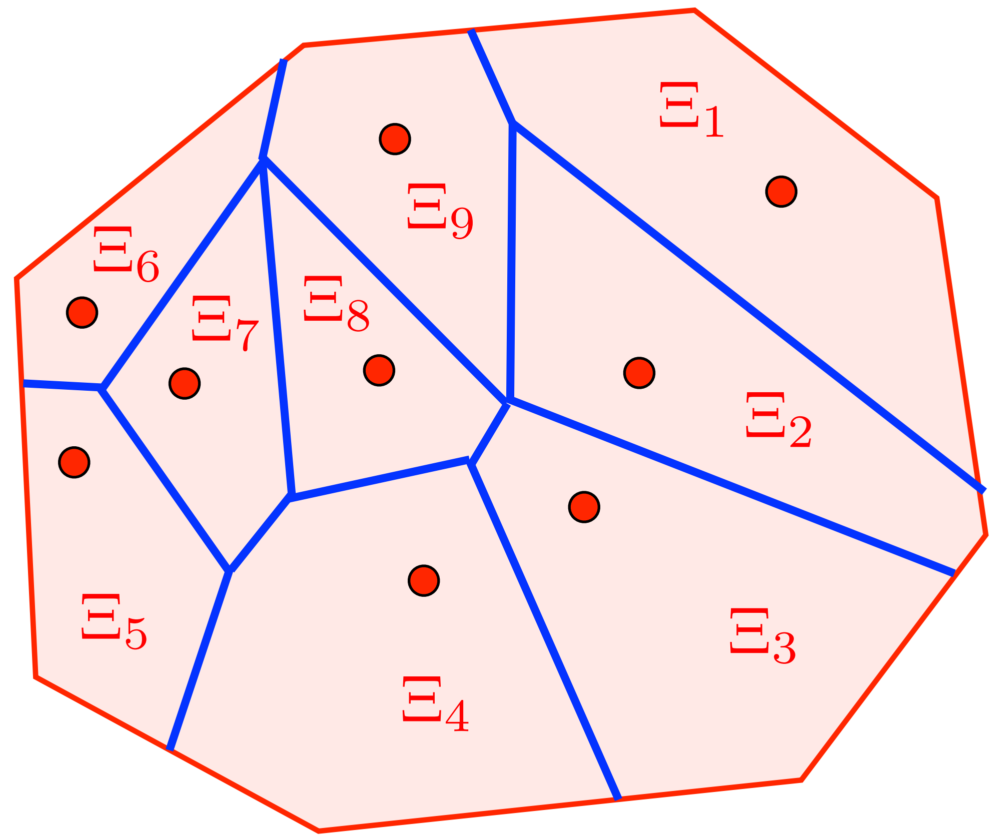
Every partition-pair (Ξ_i, Ξ_j) generates non-anticipativity constraints:

$$\Xi_i = \{\boldsymbol{\xi} \,:\, \boldsymbol{F}\boldsymbol{\xi} \leq \boldsymbol{f}\}$$

$$\Xi_j = \{\boldsymbol{\xi} \,:\, \boldsymbol{G}\boldsymbol{\xi} \leq \boldsymbol{g}\}$$

Iterative Partitioning: Voronoi Partitioning

Recall the Voronoi partitioning of uncertainty sets:



Every partition-pair (Ξ_i, Ξ_j) generates non-anticipativity constraints:

The stage-*t* decisions need to be the same across Ξ_i and Ξ_j whenever the problem

$$\left.\begin{aligned} \Xi_i &= \{\boldsymbol{\xi} \, : \, \boldsymbol{F}\boldsymbol{\xi} \leq \boldsymbol{f}\} \\ \Xi_j &= \{\boldsymbol{\xi} \, : \, \boldsymbol{G}\boldsymbol{\xi} \leq \boldsymbol{g}\} \end{aligned}\right\}$$

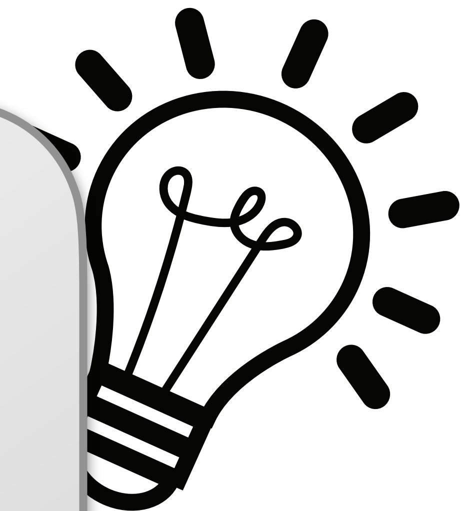
$$\begin{array}{ll} \underset{\boldsymbol{\xi}^i, \boldsymbol{\xi}^j, \alpha, \beta}{\text{maximize}} & \alpha + \beta \\ \text{subject to} & \boldsymbol{F}\boldsymbol{\xi}^i + \alpha \mathbf{e} \leq \boldsymbol{f}, \quad \boldsymbol{G}\boldsymbol{\xi}^j + \beta \mathbf{e} \leq \boldsymbol{g} \\ & (\boldsymbol{\xi}^i_1, \ldots, \boldsymbol{\xi}^i_t) = (\boldsymbol{\xi}^j_1, \ldots, \boldsymbol{\xi}^j_t) \end{array}$$

is feasible with strictly positive value.

Iterative Partitioning: Voronoi Partitioning

Recall the Voronoi partitioning of uncertainty sets:

Intuition



- The objective function tries to determine two realizations $\boldsymbol{\xi}^i \in \operatorname{int} \Xi_i$ and $\boldsymbol{\xi}^j \in \operatorname{int} \Xi_j$: by construction, they will thus be different

- The second constraint ensures that $\boldsymbol{\xi}^i$ and $\boldsymbol{\xi}^j$ are indistinguishable at time *t*

Every partition-pair (Ξ_i, Ξ_j) generates non-anticipativity constraints:

$$\left.\begin{aligned} \Xi_i &= \{\boldsymbol{\xi} \,:\, \boldsymbol{F}\boldsymbol{\xi} \leq \boldsymbol{f}\} \\ \Xi_j &= \{\boldsymbol{\xi} \,:\, \boldsymbol{G}\boldsymbol{\xi} \leq \boldsymbol{g}\} \end{aligned}\right\}$$

The stage-*t* decisions need to be the same across Ξ_i and Ξ_j whenever the problem

$$\begin{aligned} \underset{\boldsymbol{\xi}^i, \boldsymbol{\xi}^j, \alpha, \beta}{\text{maximize}} \quad & \alpha + \beta \\ \text{subject to} \quad & \boldsymbol{F}\boldsymbol{\xi}^i + \alpha \mathrm{e} \leq \boldsymbol{f}, \quad \boldsymbol{G}\boldsymbol{\xi}^j + \beta \mathrm{e} \leq \boldsymbol{g} \\ & (\boldsymbol{\xi}^i_1, \ldots, \boldsymbol{\xi}^i_t) = (\boldsymbol{\xi}^j_1, \ldots, \boldsymbol{\xi}^j_t) \end{aligned}$$

is feasible with strictly positive value.

Part 3 **Continuous Recourse Decisions**

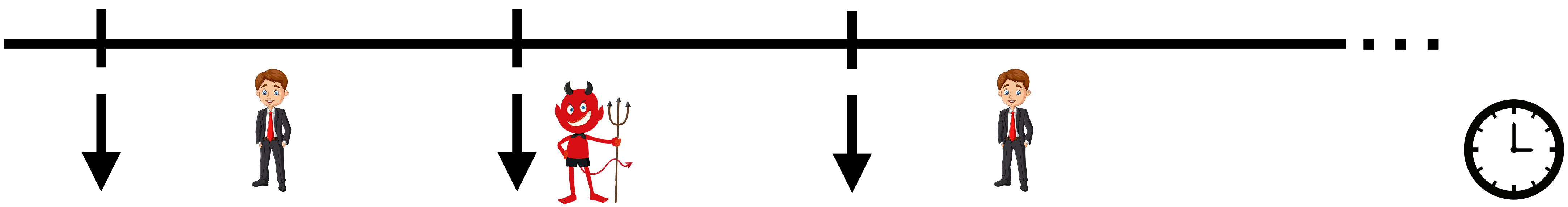
Multi-Stage Models

- Time (In-)Consistency
- Decision Rules
- Iterative Partitioning
- **Nested Benders' Decomposition**
- Robust Dual Dynamic Programming

Georghiou, Tsoukalas and W. *Robust Dual Dynamic Programming*. OR, 2019.

Nested Benders' Decomposition

Recall the multi-stage robust optimization problem:

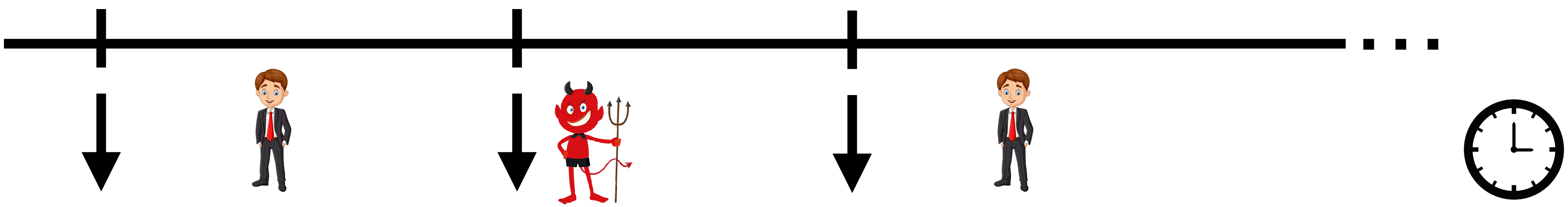


$$\begin{array}{ll} \min & \boldsymbol{q}_1^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1 \end{array} + \max_{\boldsymbol{\xi}_2 \in \Xi_2} \begin{array}{ll} \min & \boldsymbol{q}_2^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{x}_2 \in \mathcal{X}_2 \end{array} + \cdots$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}_2)\,\boldsymbol{x}_1 + \boldsymbol{W}_2\,\boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}_2)$$

Nested Benders' Decomposition

Recall the multi-stage robust optimization problem:



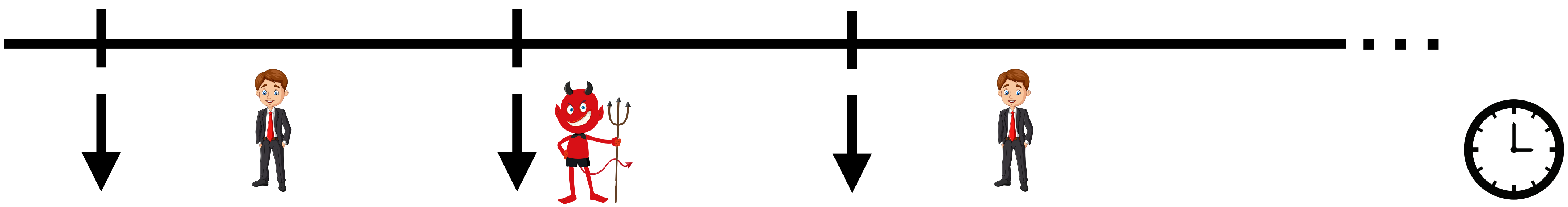
$$\begin{array}{ll}\min & \boldsymbol{q}_1^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1\end{array} + \max_{\boldsymbol{\xi}_2 \in \Xi_2} \begin{array}{ll}\min & \boldsymbol{q}_2^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{x}_2 \in \mathcal{X}_2\end{array} + \cdots$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}_2)\,\boldsymbol{x}_1 + \boldsymbol{W}_2\,\boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}_2)$$

$$\boldsymbol{x}_2 \in \mathcal{X}_2(\boldsymbol{x}_1, \boldsymbol{\xi}_2)$$

Nested Benders' Decomposition

Recall the multi-stage robust optimization problem:



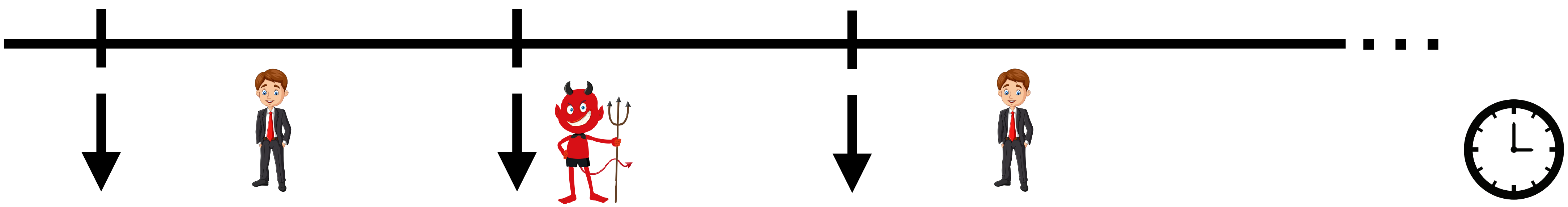
$$\begin{array}{ll} \min & \boldsymbol{q}_1^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1 \end{array} + \max_{\boldsymbol{\xi}_2 \in \Xi_2} \begin{array}{ll} \min & \boldsymbol{q}_2^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{x}_2 \in \mathcal{X}_2 \end{array} + \mathcal{Q}_3(\boldsymbol{x}_2)$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}_2)\,\boldsymbol{x}_1 + \boldsymbol{W}_2\,\boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}_2)$$

$$\boldsymbol{x}_2 \in \mathcal{X}_2(\boldsymbol{x}_1, \boldsymbol{\xi}_2)$$

Nested Benders' Decomposition

Recall the multi-stage robust optimization problem:



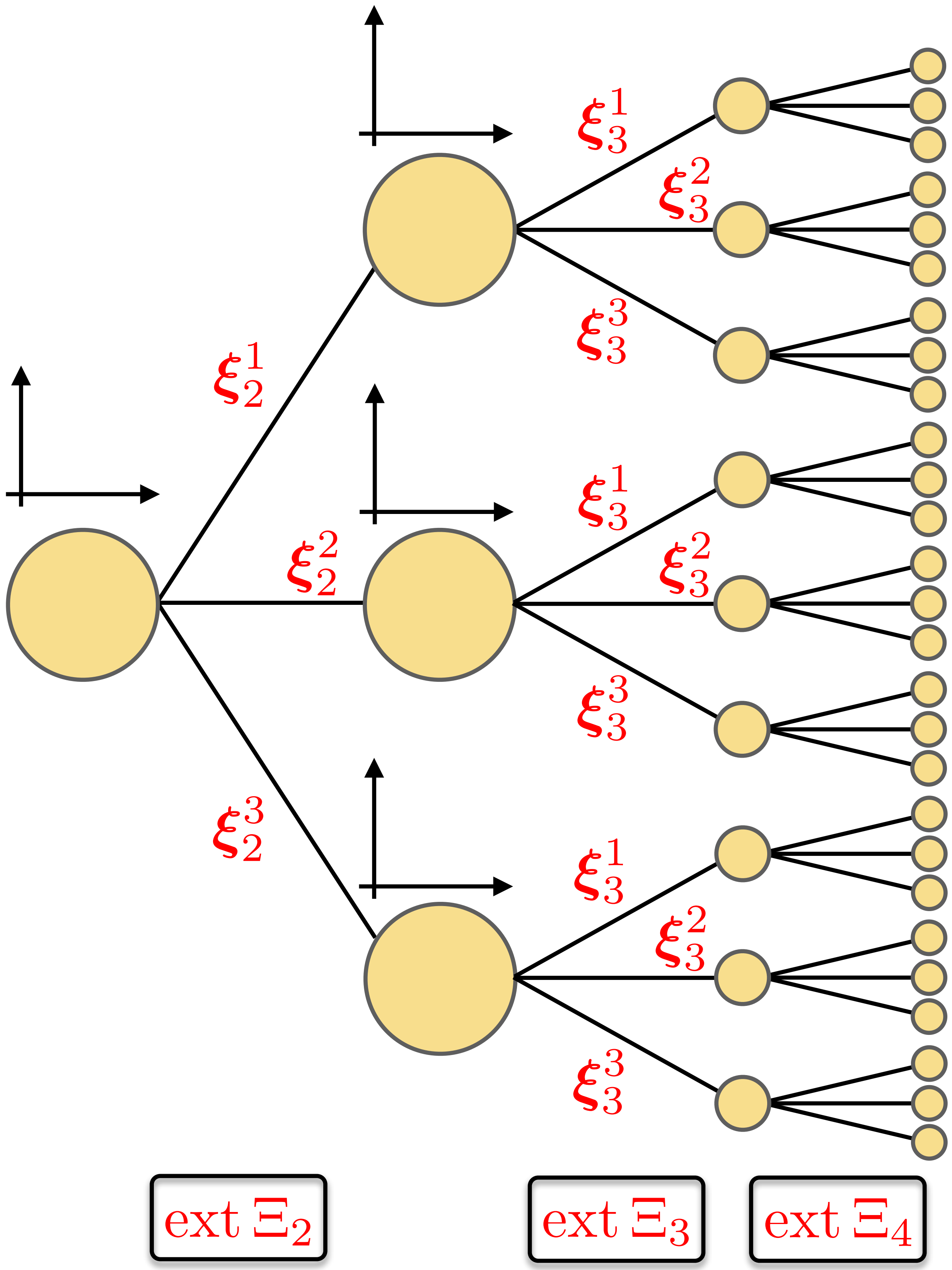
$$\begin{array}{ll} \min & \boldsymbol{q}_1^\top \boldsymbol{x}_1 \\ \text{s.t.} & \boldsymbol{x}_1 \in \mathcal{X}_1 \end{array} + \max_{\boldsymbol{\xi}_2 \in \Xi_2} \begin{array}{ll} \min & \boldsymbol{q}_2^\top \boldsymbol{x}_2 \\ \text{s.t.} & \boldsymbol{x}_2 \in \mathcal{X}_2 \end{array} + \mathcal{Q}_3(\boldsymbol{x}_2)$$

$$\boldsymbol{T}_2(\boldsymbol{\xi}_2)\,\boldsymbol{x}_1 + \boldsymbol{W}_2\,\boldsymbol{x}_2 \geq \boldsymbol{h}_2(\boldsymbol{\xi}_2)$$

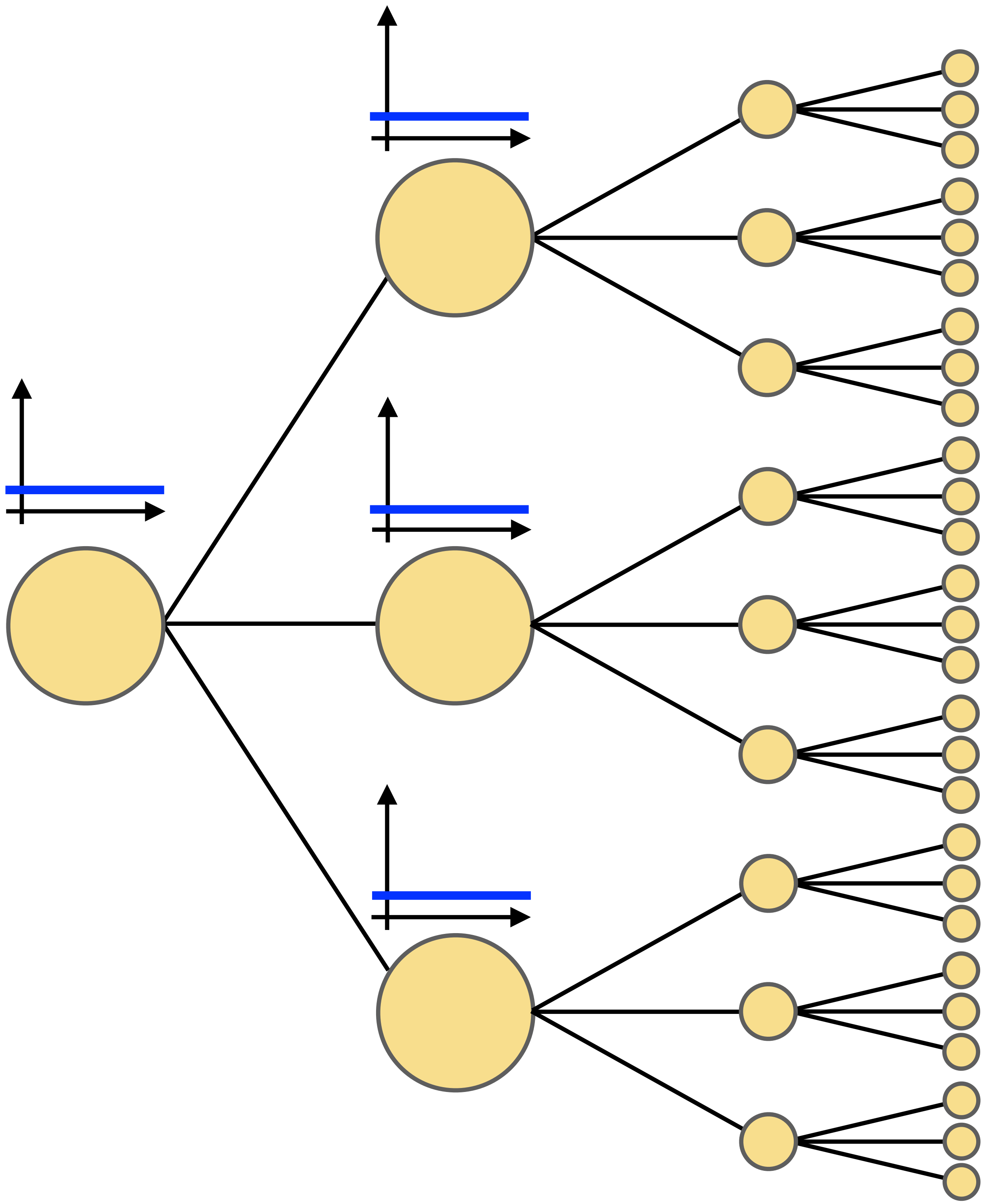
$$\boldsymbol{x}_2 \in \mathcal{X}_2(\boldsymbol{x}_1, \boldsymbol{\xi}_2)$$

stage-wise rectangularity: $\Xi = \Xi_1 \times \ldots \times \Xi_T$

Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ on worst-case costs to-go

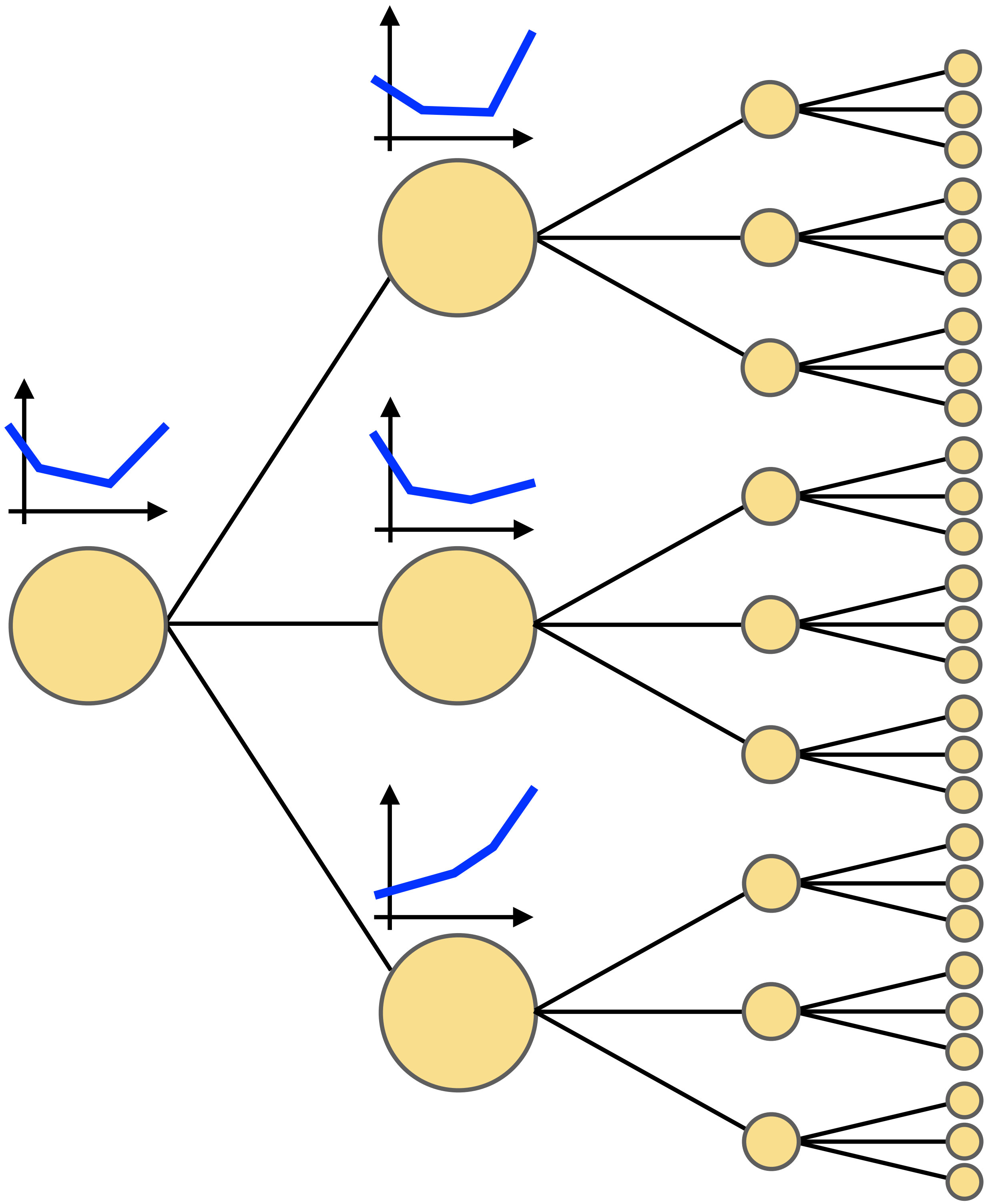


Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ on worst-case costs to-go



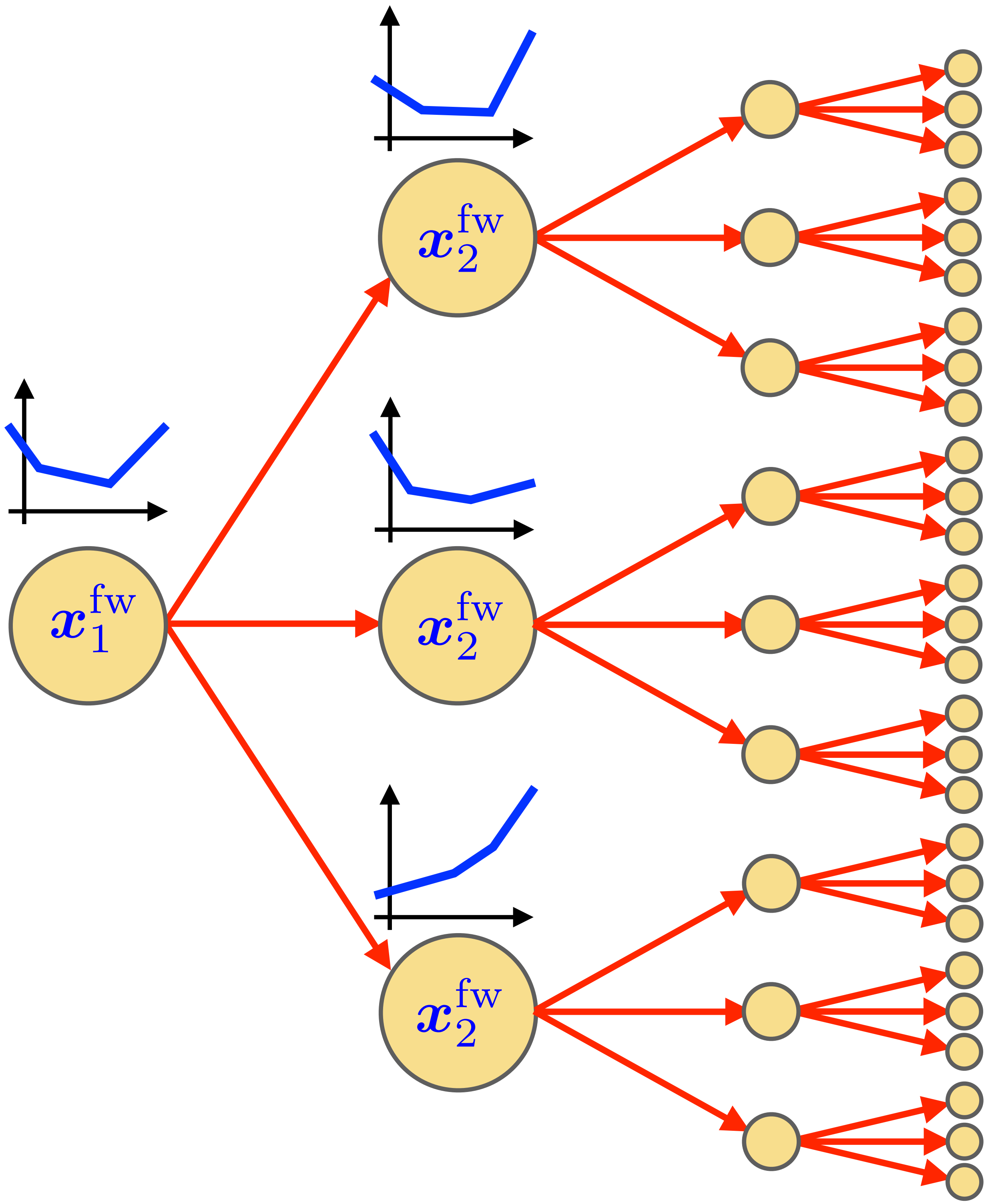
1 **Initialization.** Set all lower bounds to $-\infty$.

Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ on worst-case costs to-go



1 **Initialization.** Set all lower bounds to $-\infty$.

Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ on worst-case costs to-go



1. **Initialization.** Set all lower bounds to $-\infty$.

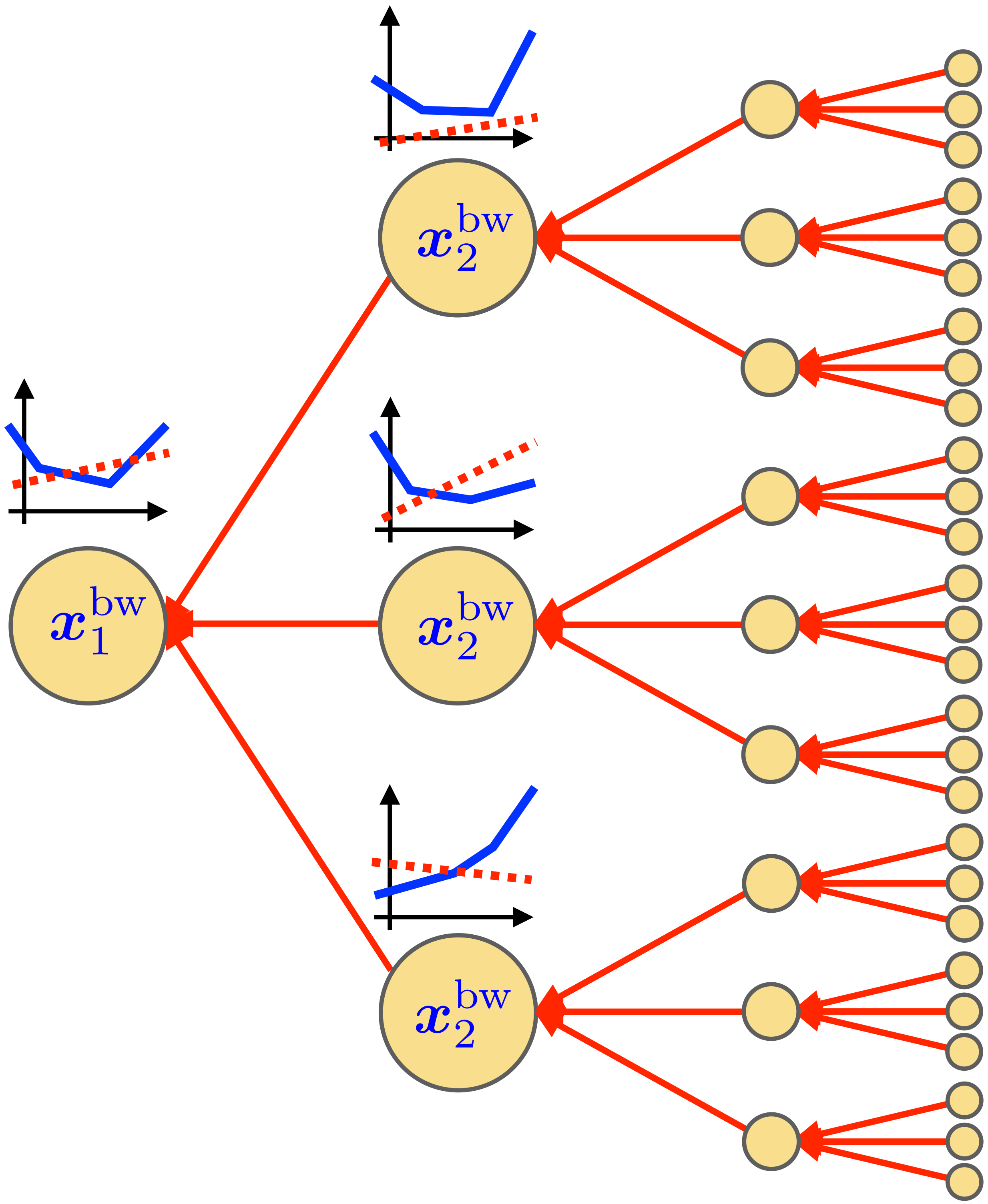
2. **Forward Pass.** Find nodal solutions $\boldsymbol{x}_t^{\mathrm{fw}}$ optimizing

$$\min_{\boldsymbol{x}_t \in \mathcal{X}_t(\boldsymbol{x}_{t-1}^{\mathrm{fw}}, \boldsymbol{\xi}_t)} \quad \boldsymbol{q}_t{}^\top \boldsymbol{x}_t + \underline{\mathcal{Q}}_{t+1}(\boldsymbol{x}_t)$$

for all t = 1, …, T.

Nested Benders' Decomposition

Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ on worst-case costs to-go



1. **Initialization.** Set all lower bounds to $-\infty$.

2. **Forward Pass.** Find nodal solutions $\boldsymbol{x}_t^{\mathrm{fw}}$ optimizing

$$\min_{\boldsymbol{x}_t \in \mathcal{X}_t(\boldsymbol{x}_{t-1}^{\mathrm{fw}}, \boldsymbol{\xi}_t)} \quad \boldsymbol{q}_t{}^\top \boldsymbol{x}_t + \underline{\mathcal{Q}}_{t+1}(\boldsymbol{x}_t)$$

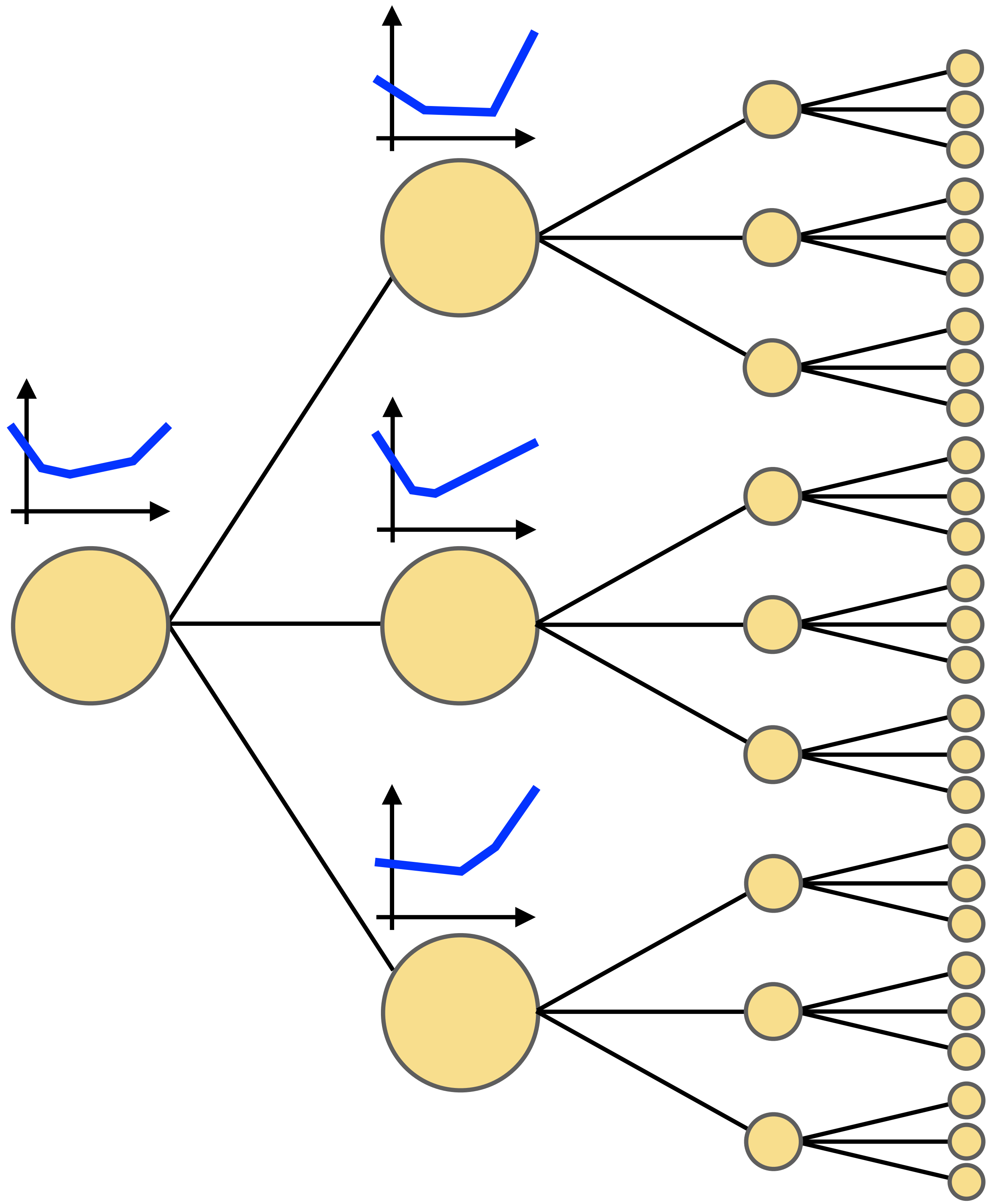
for all $t = 1, \ldots, T$.

3. **Backward Pass.** Find nodal solutions $\boldsymbol{x}_t^{\mathrm{bw}}$ optimizing

$$\min_{\boldsymbol{x}_t \in \mathcal{X}_t(\boldsymbol{x}_{t-1}^{\mathrm{fw}}, \boldsymbol{\xi}_t)} \quad \boldsymbol{q}_t{}^\top \boldsymbol{x}_t + \underline{\mathcal{Q}}_{t+1}(\boldsymbol{x}_t)$$

for all $t = T, \ldots, 1$. Use dual information to update $\underline{\mathcal{Q}}_t$.

Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ on worst-case costs to-go



1. **Initialization.** Set all lower bounds to $-\infty$.

2. **Forward Pass.** Find nodal solutions $\boldsymbol{x}_t^{\mathrm{fw}}$ optimizing

$$\min_{\boldsymbol{x}_t \in \mathcal{X}_t(\boldsymbol{x}_{t-1}^{\mathrm{fw}}, \boldsymbol{\xi}_t)} \quad \boldsymbol{q}_t{}^\top \boldsymbol{x}_t + \underline{\mathcal{Q}}_{t+1}(\boldsymbol{x}_t)$$

for all $t = 1, \ldots, T$.

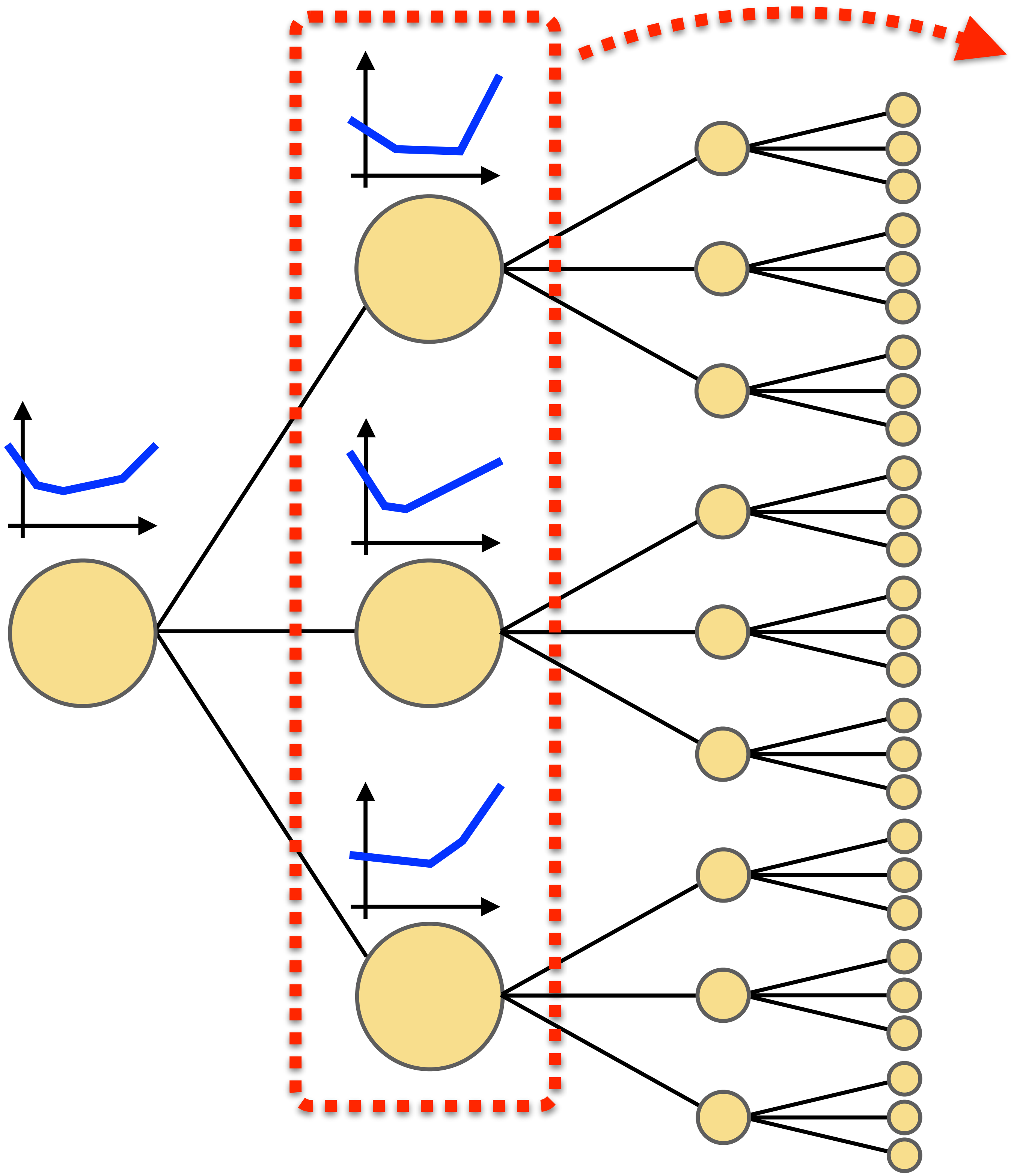
3. **Backward Pass.** Find nodal solutions $\boldsymbol{x}_t^{\mathrm{bw}}$ optimizing

$$\min_{\boldsymbol{x}_t \in \mathcal{X}_t(\boldsymbol{x}_{t-1}^{\mathrm{fw}}, \boldsymbol{\xi}_t)} \quad \boldsymbol{q}_t{}^\top \boldsymbol{x}_t + \underline{\mathcal{Q}}_{t+1}(\boldsymbol{x}_t)$$

for all $t = T, \ldots, 1$. Use dual information to update $\underline{\mathcal{Q}}_t$.

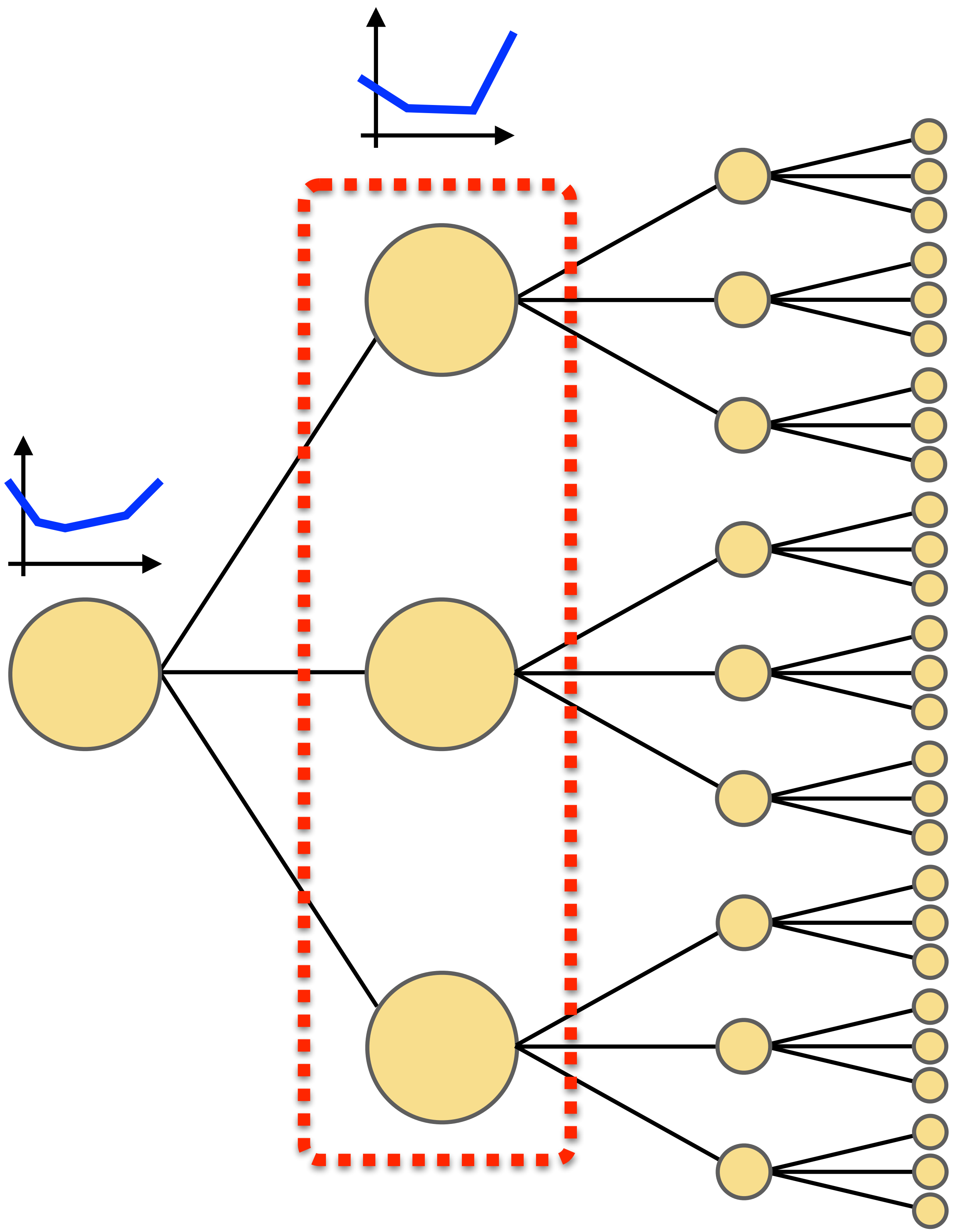
Repeat until bounds don't change.

Nested Benders' Decomposition: Cut Sharing



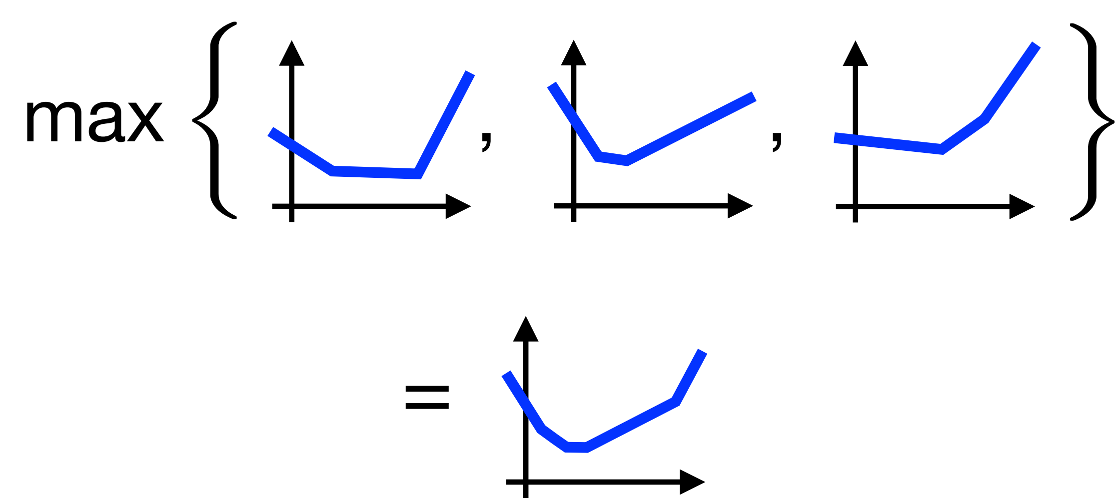
The worst-case costs to-go $\mathcal{Q}_t$ only depend on $\boldsymbol{x}_{t-1}$, not on the parameter trajectory.

Nested Benders' Decomposition: Cut Sharing

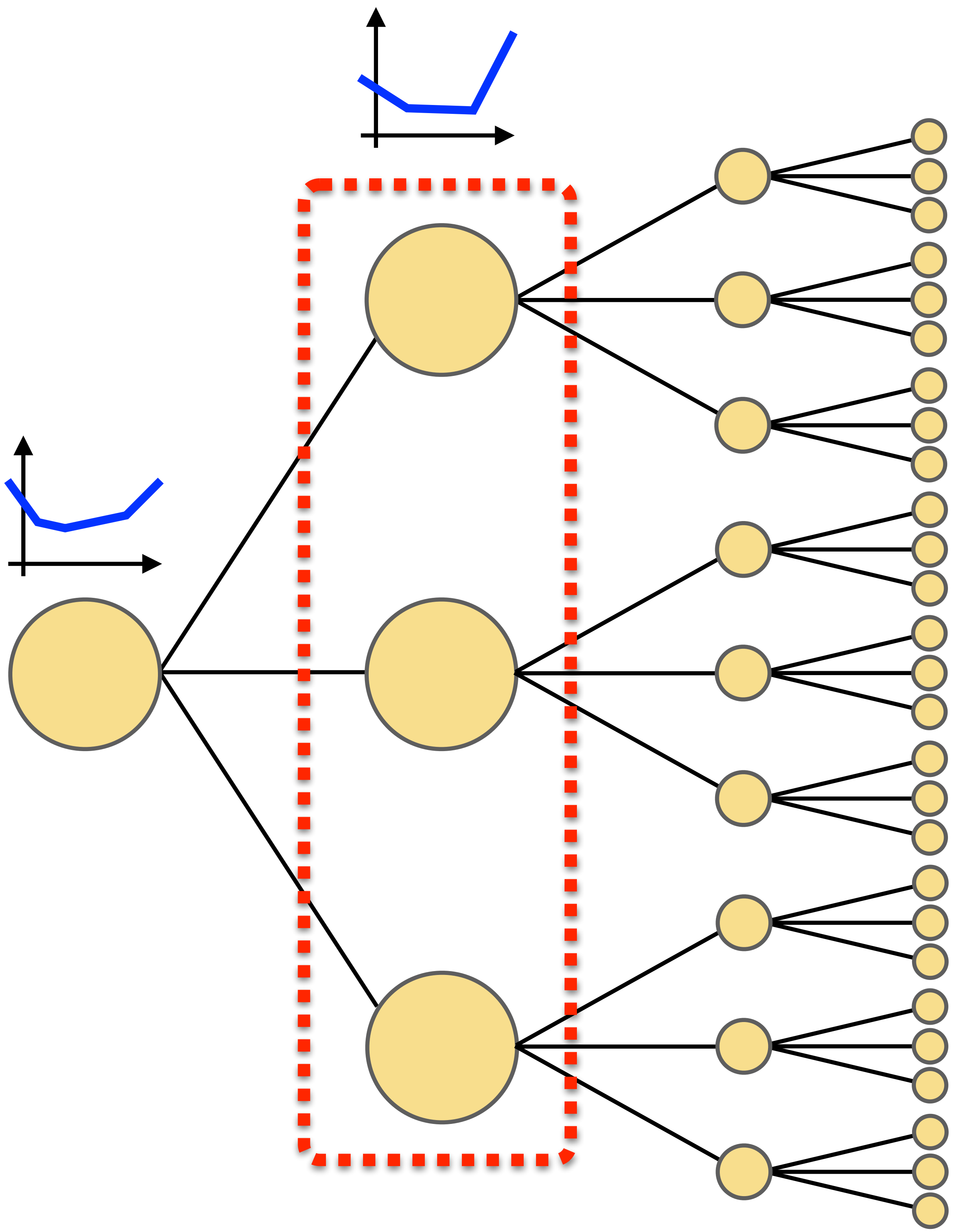


The worst-case costs to-go $\mathcal{Q}_t$ only depend on $\boldsymbol{x}_{t-1}$, not on the parameter trajectory.

The cuts in each time stage can be shared across all nodes in that stage!



Nested Benders' Decomposition: Cut Sharing

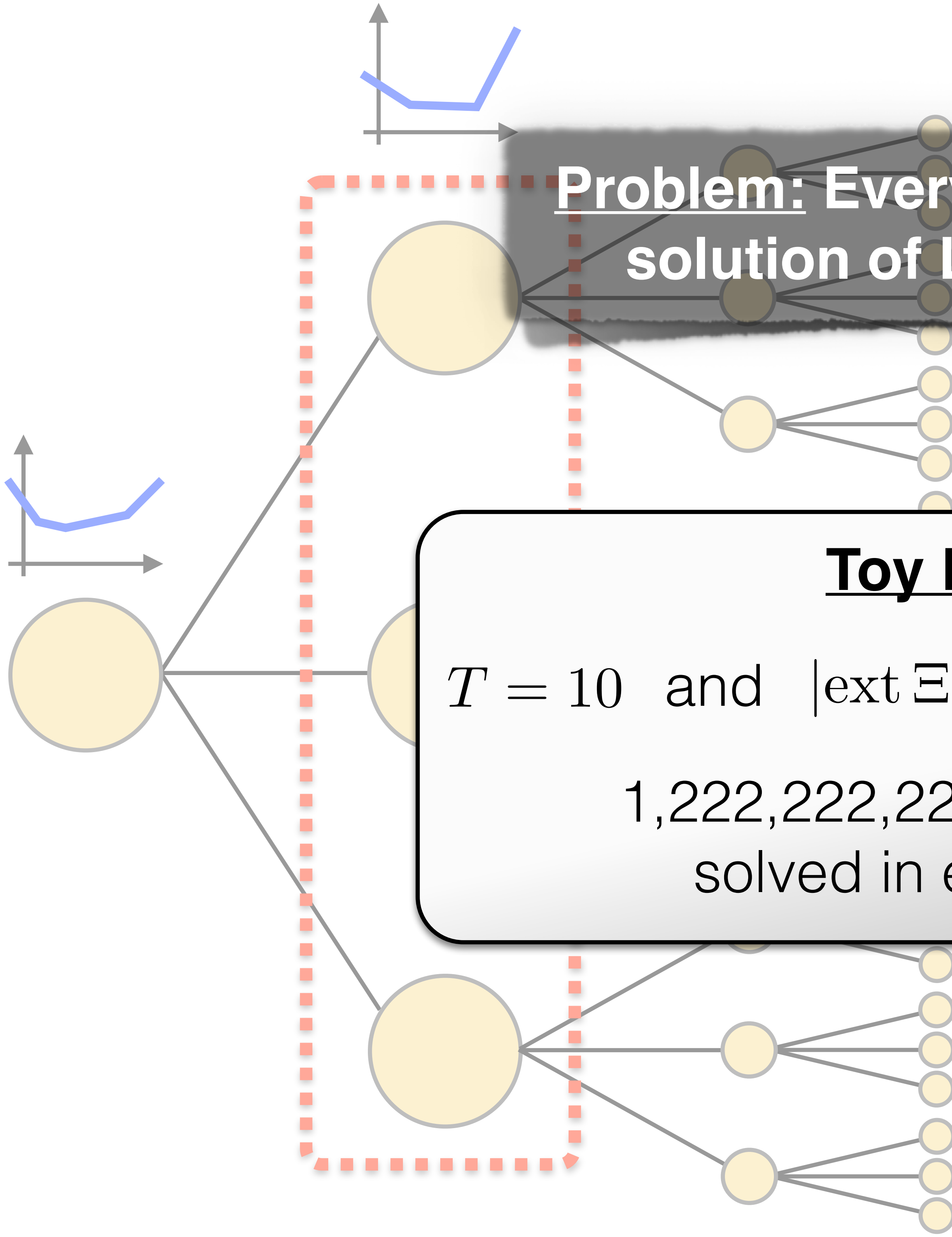


The worst-case costs to-go $\mathcal{Q}_t$ only depend on $\boldsymbol{x}_{t-1}$, not on the parameter trajectory.

The cuts in each time stage can be shared across all nodes in that stage!

- lower memory requirements
- faster convergence
- still need to solve all nodal problems

Nested Benders' Decomposition: Cut Sharing



The worst-case costs to-go Q_t only depend on x_{t-1}, not on the parameter trajectory.

Problem: Every iteration requires solution of LPs at all nodes!

time stage
across all
stage!

irements

faster c

still need to solve all nodal problems

Toy Example:

$$T = 10 \quad \text{and} \quad |\text{ext}\,\Xi_2| = \ldots = |\text{ext}\,\Xi_T| = 10:$$

1,222,222,220 LPs need to be solved in every iteration!

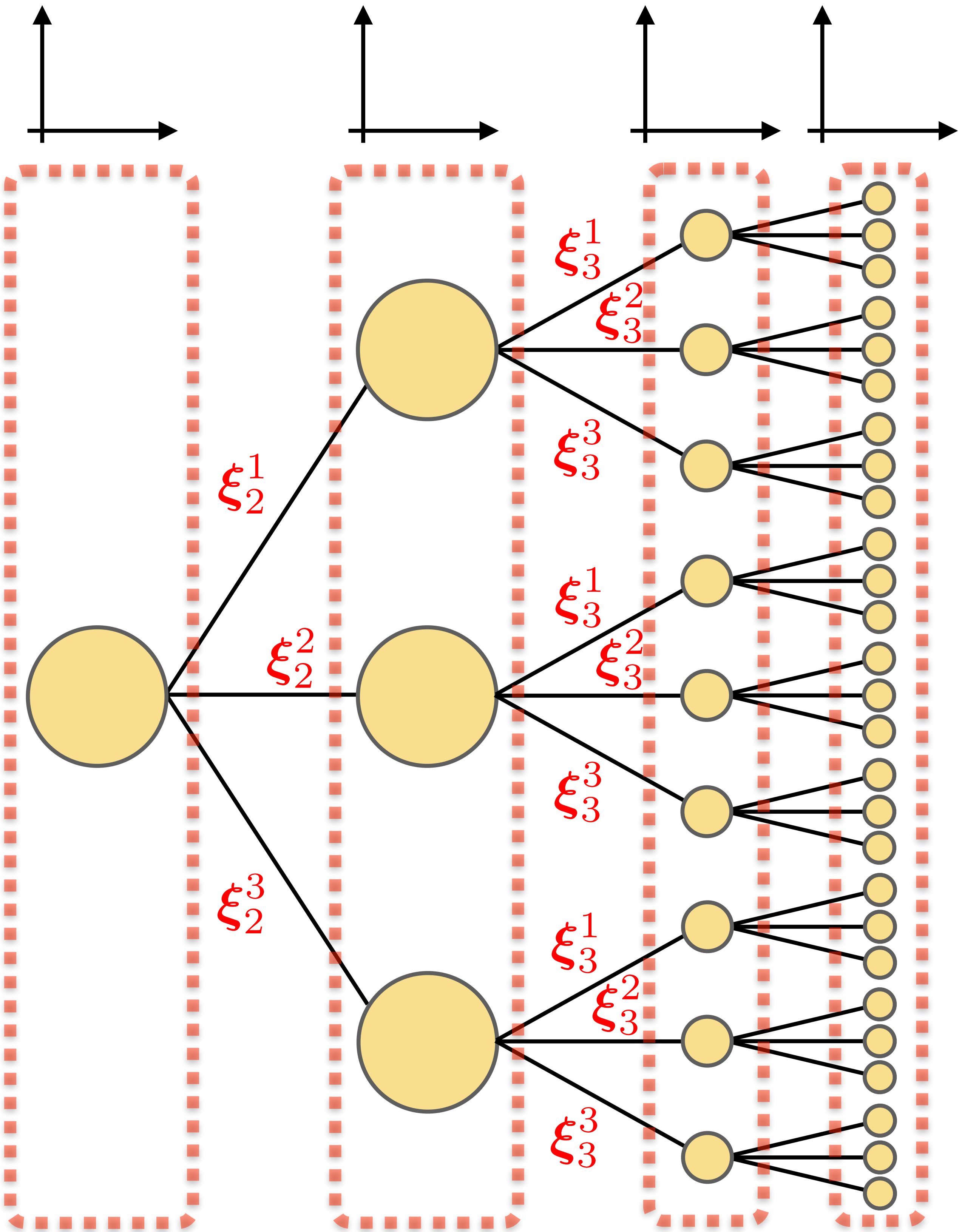
Part 3 **Continuous Recourse Decisions**

Multi-Stage Models

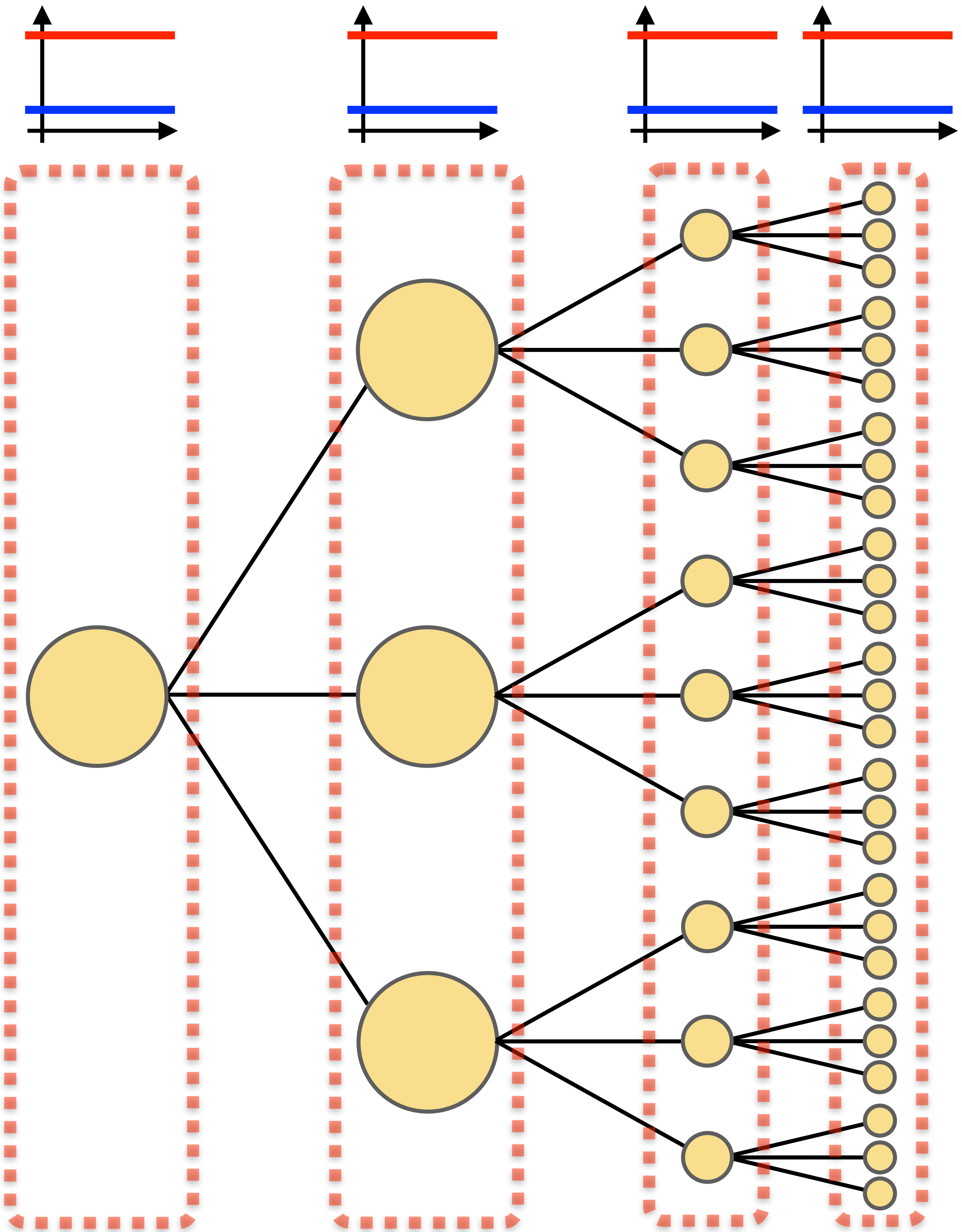
- Time (In-)Consistency
- Decision Rules
- Iterative Partitioning
- Nested Benders' Decomposition
- **Robust Dual Dynamic Programming**

Georghiou, Tsoukalas and W. *Robust Dual Dynamic Programming*. OR, 2019.

Algorithm: Refine lower bounds $\underline{Q}_t$ ***and*** upper bounds $\overline{Q}_t$

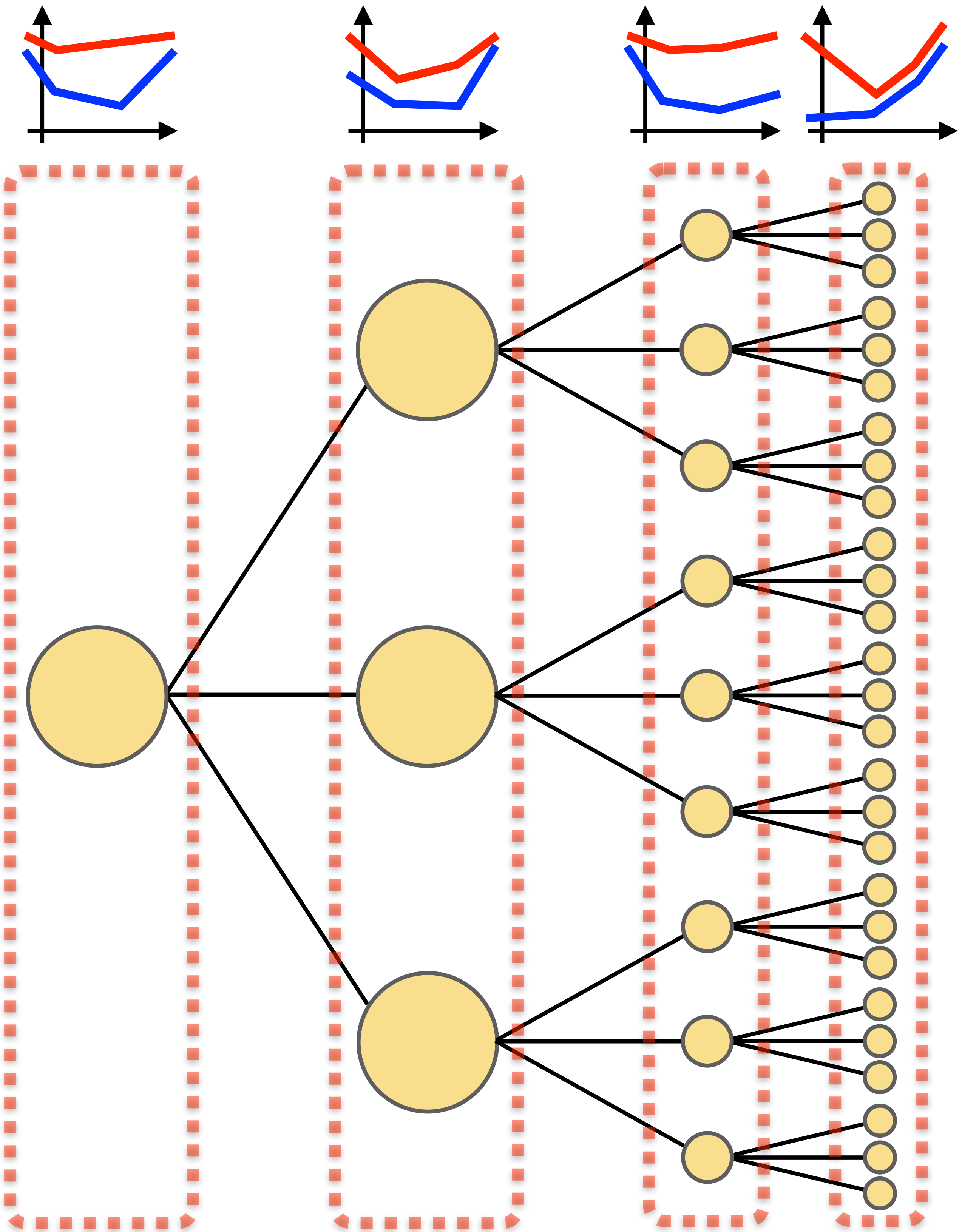


Algorithm: Refine lower bounds $\underline{Q}_t$ ***and*** upper bounds $\overline{Q}_t$



1 **Initialization.** Set all bounds to $+/-\infty$.

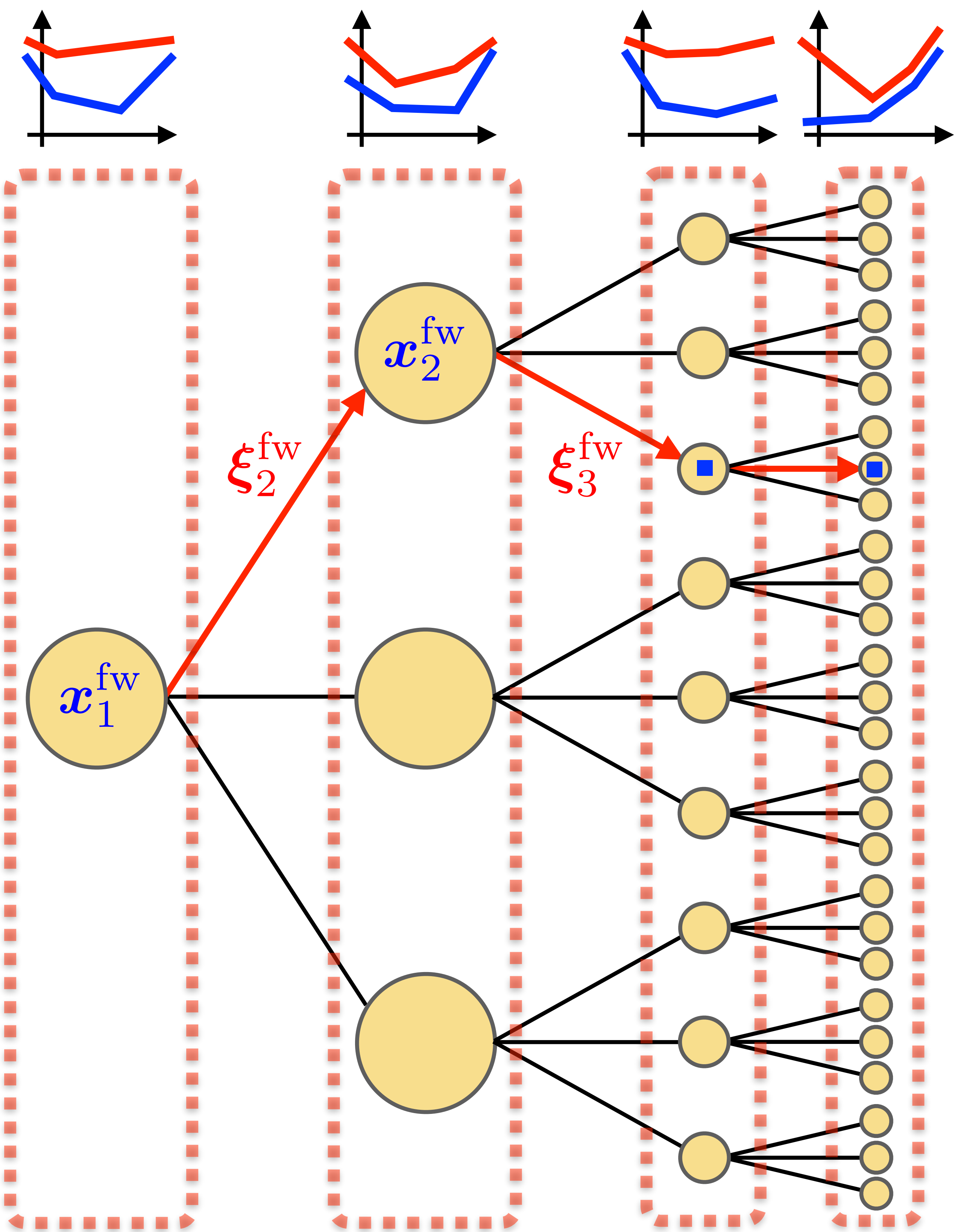
Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ ***and*** upper bounds $\overline{\mathcal{Q}}_t$



1. **Initialization.** Set all bounds to $+/-\infty$.

Robust Dual Dynamic Programming

Algorithm: Refine lower bounds $\underline{Q}_t$ ***and*** upper bounds $\overline{Q}_t$



1. **Initialization.** Set all bounds to +/–∞.

2. **Forward Pass.** Find worst-case scenarios ξ_t^{fw} optimizing

$$\max_{\xi_t \in \Xi_t} \min_{x_t \in \mathcal{X}_t(x_{t-1}^{\text{fw}}, \xi_t)} q_t^\top x_t + \overline{Q}_{t+1}(x_t)$$

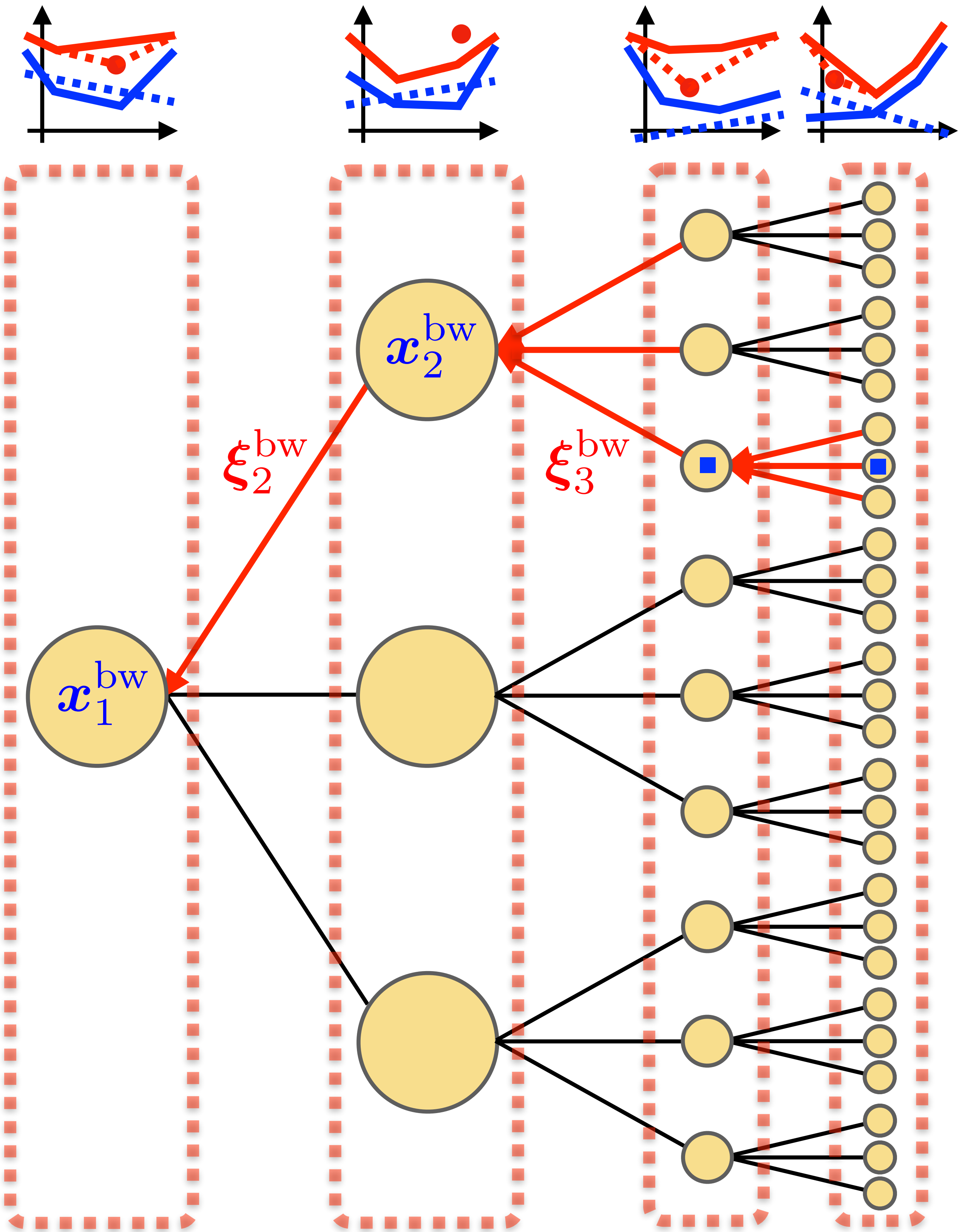
and decisions x_t^{fw} optimizing

$$\min_{x_t \in \mathcal{X}_t(x_{t-1}^{\text{fw}}, \xi_t^{\text{fw}})} q_t^\top x_t + \underline{Q}_{t+1}(x_t)$$

for all *t* = 1, …, *T* for one parameter trajectory in the tree.

Robust Dual Dynamic Programming

Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ ***and*** upper bounds $\overline{\mathcal{Q}}_t$



1. **Initialization.** Set all bounds to +/–∞.
2. **Forward Pass.** (…)
3. **Backward Pass.** Find worst-case scenarios $\boldsymbol{\xi}_t^{\mathrm{bw}}$ optimizing

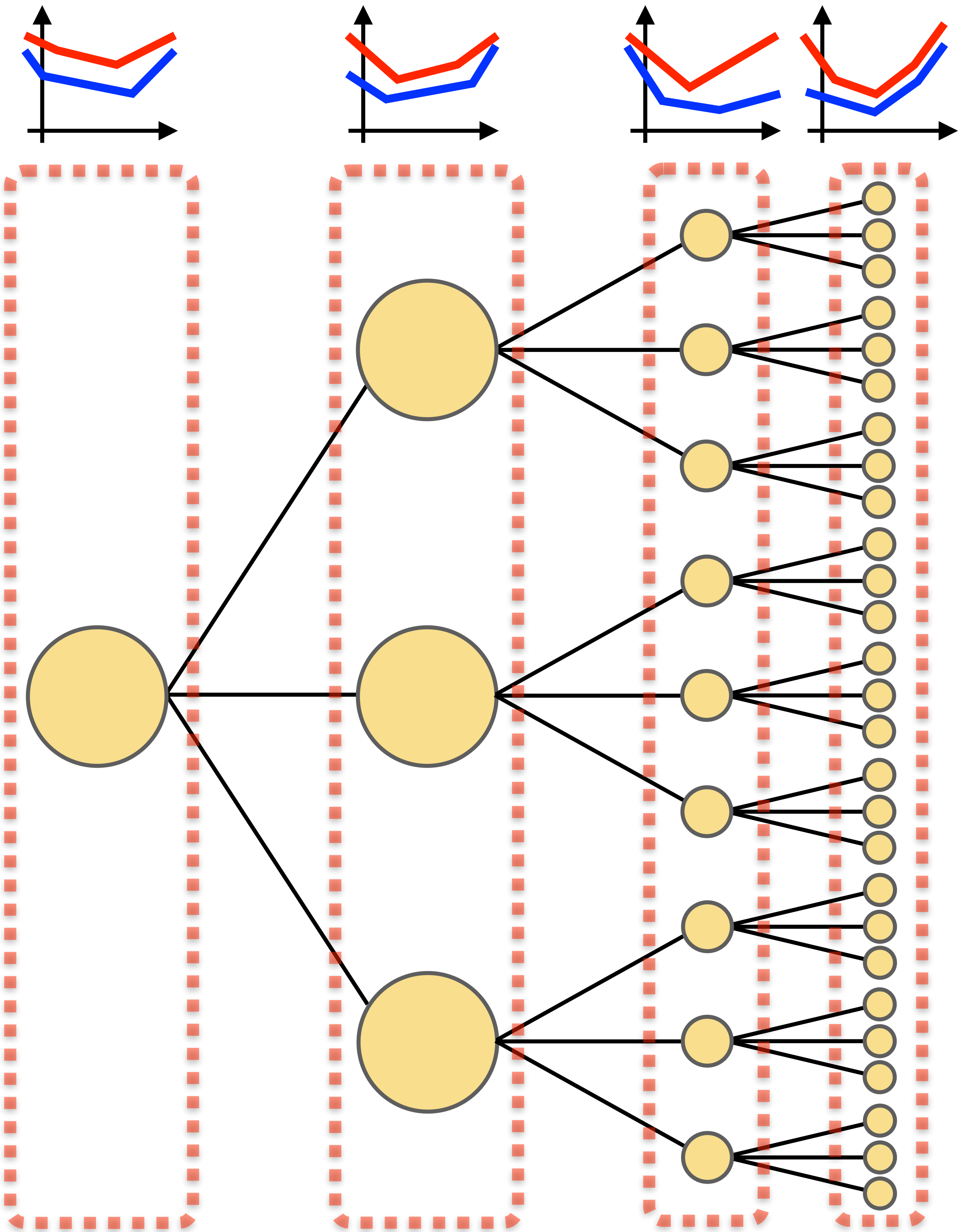
$$\max_{\boldsymbol{\xi}_t \in \Xi_t} \min_{\boldsymbol{x}_t \in \mathcal{X}_t(\boldsymbol{x}_{t-1}^{\mathrm{fw}}, \boldsymbol{\xi}_t)} \boldsymbol{q}_t^\top \boldsymbol{x}_t + \overline{\mathcal{Q}}_{t+1}(\boldsymbol{x}_t)$$

and decisions $\boldsymbol{x}_t^{\mathrm{bw}}$ optimizing

$$\min_{\boldsymbol{x}_t \in \mathcal{X}_t(\boldsymbol{x}_{t-1}^{\mathrm{fw}}, \boldsymbol{\xi}_t^{\mathrm{bw}})} \boldsymbol{q}_t^\top \boldsymbol{x}_t + \underline{\mathcal{Q}}_{t+1}(\boldsymbol{x}_t)$$

along that trajectory. Use first problem to update $\overline{\mathcal{Q}}_t$ and second on to update $\underline{\mathcal{Q}}_t$.

Algorithm: Refine lower bounds $\underline{\mathcal{Q}}_t$ ***and*** upper bounds $\overline{\mathcal{Q}}_t$



1. **Initialization.** Set all bounds to $+/-\infty$.
2. **Forward Pass.** (…)
3. **Backward Pass.** (…)

Repeat until first-stage solution $\boldsymbol{x}_1^{\mathrm{fw}}$ satisfies

$$\underline{\mathcal{Q}}_2(\boldsymbol{x}_1^{\mathrm{fw}}) = \overline{\mathcal{Q}}_2(\boldsymbol{x}_1^{\mathrm{fw}});$$

this solution is guaranteed to be optimal.

Part 4 **Discrete Recourse Decisions**

- **Decision Rules**
- Iterative Partitioning
- *K*-Adaptability

Bertsimas and Caramanis. *Adaptability via Sampling*. IEEE CDC, 2007.
Bertsimas and Georghiou. *Design of Near Optimal Decision Rules in Multistage Adaptive Mixed-Integer Optimization*. OR, 2015.
Bertsimas and Georghiou. *Binary Decision Rules for Multistage Adaptive Mixed-Integer Optimization*. MP, 2018.

Decision rule formulation of the two-stage RO problem with mixed-integer recourse:

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X},\ \boxed{\boldsymbol{y} : \Xi \mapsto \mathcal{Y}},\ \boxed{\boldsymbol{z} : \Xi \mapsto \mathbb{Z}} & \end{array}$$

continuous

discrete

Decision rule formulation of the two-stage RO problem with mixed-integer recourse:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \mathbb{Z} \end{array}$$

1 **Use affine decision rules for the recourse decisions:**

$$\begin{array}{ll} \underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),(\boldsymbol{Z},\boldsymbol{z})}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\left[\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}\right] + \\ & \qquad \boldsymbol{W}\left[\boldsymbol{Z}\boldsymbol{\xi} + \boldsymbol{z}\right] \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y},\boldsymbol{y}), \quad (\boldsymbol{Z},\boldsymbol{z}) \end{array}$$

Decision rule formulation of the two-stage RO problem with mixed-integer recourse:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \mathbb{Z} \end{array}$$

2 **Enforce integrality of integer decisions:**

$$\begin{array}{ll} \underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),(\boldsymbol{Z},\boldsymbol{z})}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,[\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] + \\ & \qquad\qquad \boldsymbol{W}\,[\boldsymbol{Z}\lceil\boldsymbol{\xi}\rceil + \boldsymbol{z}] \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y},\boldsymbol{y}), \quad (\boldsymbol{Z},\boldsymbol{z}) \text{ integer} \end{array}$$

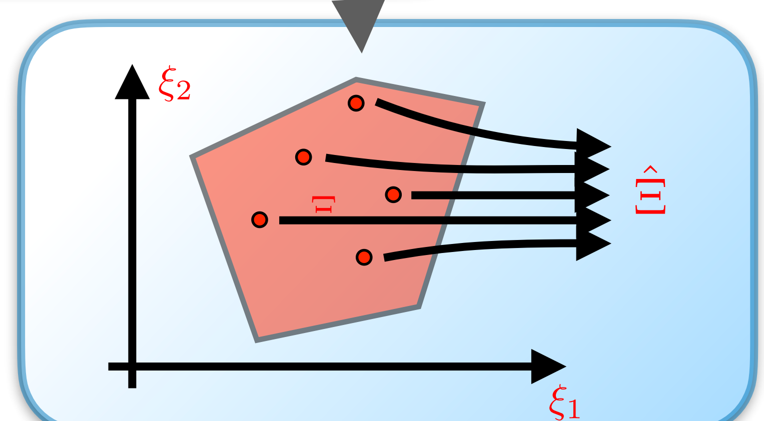
Decision Rules: Constraint Sampling

Decision rule formulation of the two-stage RO problem with mixed-integer recourse:

$$
\begin{array}{ll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \mathbb{Z}
\end{array}
$$

3 Conduct constraint sampling:

$$
\begin{array}{ll}
\underset{\boldsymbol{x},(\boldsymbol{Y},\boldsymbol{y}),(\boldsymbol{Z},\boldsymbol{z})}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,[\boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}] + \\
& \qquad \boldsymbol{W}\,[\boldsymbol{Z}\lceil\boldsymbol{\xi}\rceil + \boldsymbol{z}] \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \boxed{\hat{\Xi}} \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y},\boldsymbol{y}), \quad (\boldsymbol{Z},\boldsymbol{z}) \text{ integer}
\end{array}
$$



Decision rule formulation **of the two-stage RO problem with** **mixed-integer recourse**:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \; \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \; \boldsymbol{z} : \Xi \mapsto \mathbb{Z} \end{array}$$

continuous

discrete

Decision Rules: Semi-Infinite Programming

Decision rule formulation of the two-stage RO problem with **mixed-integer recourse**:

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \mathbb{Z} &
\end{array}
$$

1 Restrict **second-stage decisions** to **piecewise affine DRs**:

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& y_i(\boldsymbol{\xi}) \equiv \max_j \{\overline{\boldsymbol{y}}_{ij}{}^\top \boldsymbol{\xi} + \overline{y}_{ij}\} - \max_j \{\underline{\boldsymbol{y}}_{ij}{}^\top \boldsymbol{\xi} + \underline{y}_{ij}\} & \forall i \\
& z_i(\boldsymbol{\xi}) = 1 \Leftrightarrow \max_j \{\overline{\boldsymbol{z}}_{ij}{}^\top \boldsymbol{\xi} + \overline{z}_{ij}\} - \max_j \{\underline{\boldsymbol{z}}_{ij}{}^\top \boldsymbol{\xi} + \underline{z}_{ij}\} \leq 0 & \forall i \\
& \boldsymbol{x} \in \mathcal{X}, \quad [(\overline{\boldsymbol{y}}_{ij}, \overline{y}_{ij}), (\underline{\boldsymbol{y}}_{ij}, \underline{y}_{ij})], \quad [(\overline{\boldsymbol{z}}_{ij}, \overline{z}_{ij}), (\underline{\boldsymbol{z}}_{ij}, \underline{z}_{ij})] &
\end{array}
$$

Decision Rules: Semi-Infinite Programming

Decision rule formulation of the two-stage RO problem with mixed-integer recourse:

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \mathbb{Z} & \end{array}$$

2 **Iterate between solving two problems:**

Decision Rules: Semi-Infinite Programming

Decision rule formulation of the two-stage RO problem with mixed-integer recourse:

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \mathbb{Z} &
\end{array}
$$

2 Iterate between solving two problems:

Master problem

Find best piecewise affine DRs for finite subset $\hat{\Xi} \subseteq \Xi$

Decision Rules: Semi-Infinite Programming

Decision rule formulation of the two-stage RO problem with mixed-integer recourse:

$$
\begin{array}{ll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \mathbb{Z}
\end{array}
$$

2 Iterate between solving two problems:

Master problem

Find best piecewise affine DRs for finite subset $\hat{\Xi} \subseteq \Xi$

Subproblem

Find worst-case $\boldsymbol{\xi} \in \Xi \setminus \hat{\Xi}$ for fixed decision rules

Decision Rules: Semi-Infinite Programming

Decision rule formulation of the two-stage RO problem with mixed-integer recourse:

$$\begin{array}{lll} \underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\ & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \mathbb{Z} & \end{array}$$

2 Iterate between solving two problems:

Master problem

Find best piecewise affine DRs for finite subset $\hat{\Xi} \subseteq \Xi$

can be formulated as an MILP

Subproblem

Find worst-case $\boldsymbol{\xi} \in \Xi \setminus \hat{\Xi}$ for fixed decision rules

can be formulated as an MILP

Decision rule formulation of the two-stage RO problem with mixed-*binary* recourse:

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X},\ \boxed{\boldsymbol{y} : \Xi \mapsto \mathcal{Y},}\ \boxed{\boldsymbol{z} : \Xi \mapsto \{0,1\}} &
\end{array}
$$

continuous

binary

Decision rule formulation of the two-stage RO problem with mixed-*binary* recourse:

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \{0,1\} &
\end{array}
$$

1 **Restrict second-stage decisions to affine functions of (lifted) features:**

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{y}(\boldsymbol{\xi}) \equiv \boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}, \quad \boxed{\boldsymbol{z}(\boldsymbol{\xi}) \equiv \boldsymbol{Z}\mathcal{L}\boldsymbol{\xi} + \boldsymbol{z}} & \\
& \mathbf{0} \leq \boldsymbol{z}(\boldsymbol{\xi}) \leq \mathbf{e} & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) \text{ real}, \quad (\boldsymbol{Z}, \boldsymbol{z}) \text{ integer} &
\end{array}
$$

Decision Rules: Lifting Approach

Decision rule formulation of the two-stage RO problem with mixed-*binary* recourse:

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \{0,1\} &
\end{array}
$$

1 **Restrict second-stage decisions to affi**

(lifted) features:

$$
\mathcal{L}_i(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if } {\boldsymbol{\alpha}_i}^\top \boldsymbol{\xi} \geq \beta_i \\ 0 & \text{otherwise} \end{cases}
$$

with $(\boldsymbol{\alpha}_i, \beta_i)$ preselected.

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{y}(\boldsymbol{\xi}) \equiv \boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}, \quad \boxed{\boldsymbol{z}(\boldsymbol{\xi}) \equiv \boldsymbol{Z}\mathcal{L}\boldsymbol{\xi} + \boldsymbol{z}} & \\
& \mathbf{0} \leq \boldsymbol{z}(\boldsymbol{\xi}) \leq \mathbf{e} & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) \text{ real}, \quad (\boldsymbol{Z}, \boldsymbol{z}) \text{ integer} &
\end{array}
$$

Decision rule formulation of the two-stage RO problem with mixed-*binary* recourse:

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \{0,1\} &
\end{array}
$$

1 **Restrict second-stage decisions to affi**

(lifted) features:

$$
\mathcal{L}_i(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha}_i{}^\top \boldsymbol{\xi} \geq \beta_i \\ 0 & \text{otherwise} \end{cases}
$$

with $(\boldsymbol{\alpha}_i, \beta_i)$ preselected.

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{y}(\boldsymbol{\xi}) \equiv \boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}, \quad \boldsymbol{z}(\boldsymbol{\xi}) & \\
& \mathbf{0} \leq \boldsymbol{z}(\boldsymbol{\xi}) \leq \mathbf{e} & \\
& \boldsymbol{x} \in \mathcal{X}, \quad (\boldsymbol{Y}, \boldsymbol{y}) \text{ real}, \quad (\boldsymbol{Z}, \boldsymbol{z}) \text{ integer} &
\end{array}
$$

polynomial-size MILP if:

$\boldsymbol{\alpha}_i = \mathbf{e}_j$ and $\Xi = [\underline{\boldsymbol{\xi}}, \overline{\boldsymbol{\xi}}]$

Decision rule formulation of the two-stage RO problem with mixed-*binary* recourse:

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y} : \Xi \mapsto \mathcal{Y}, \quad \boldsymbol{z} : \Xi \mapsto \{0,1\} &
\end{array}
$$

1 Restrict second-stage decisions to affi… (lifted) features:

$$
\mathcal{L}_i(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha}_i{}^\top \boldsymbol{\xi} \geq \beta_i \\ 0 & \text{otherwise} \end{cases}
$$

with $(\boldsymbol{\alpha}_i, \beta_i)$ preselected.

$$
\begin{array}{lll}
\underset{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} & \\
\text{subject to} & \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{V}\,\boldsymbol{y}(\boldsymbol{\xi}) + \boldsymbol{W}\,\boldsymbol{z}(\boldsymbol{\xi}) \geq \boldsymbol{h}(\boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in \Xi \\
& \boldsymbol{y}(\boldsymbol{\xi}) \equiv \boldsymbol{Y}\boldsymbol{\xi} + \boldsymbol{y}, \quad \boldsymbol{z}(\boldsymbol{\xi}) \ldots & \\
& \boldsymbol{0} \leq \boldsymbol{z}(\ldots & \\
& \boldsymbol{x} \in \mathcal{X} \ldots &
\end{array}
$$

polynomial-size MILP if:

$\boldsymbol{\alpha}_i = \mathrm{e}_j$ and $\Xi = [\underline{\boldsymbol{\xi}}, \overline{\boldsymbol{\xi}}]$

exponential-size MILP otherwise

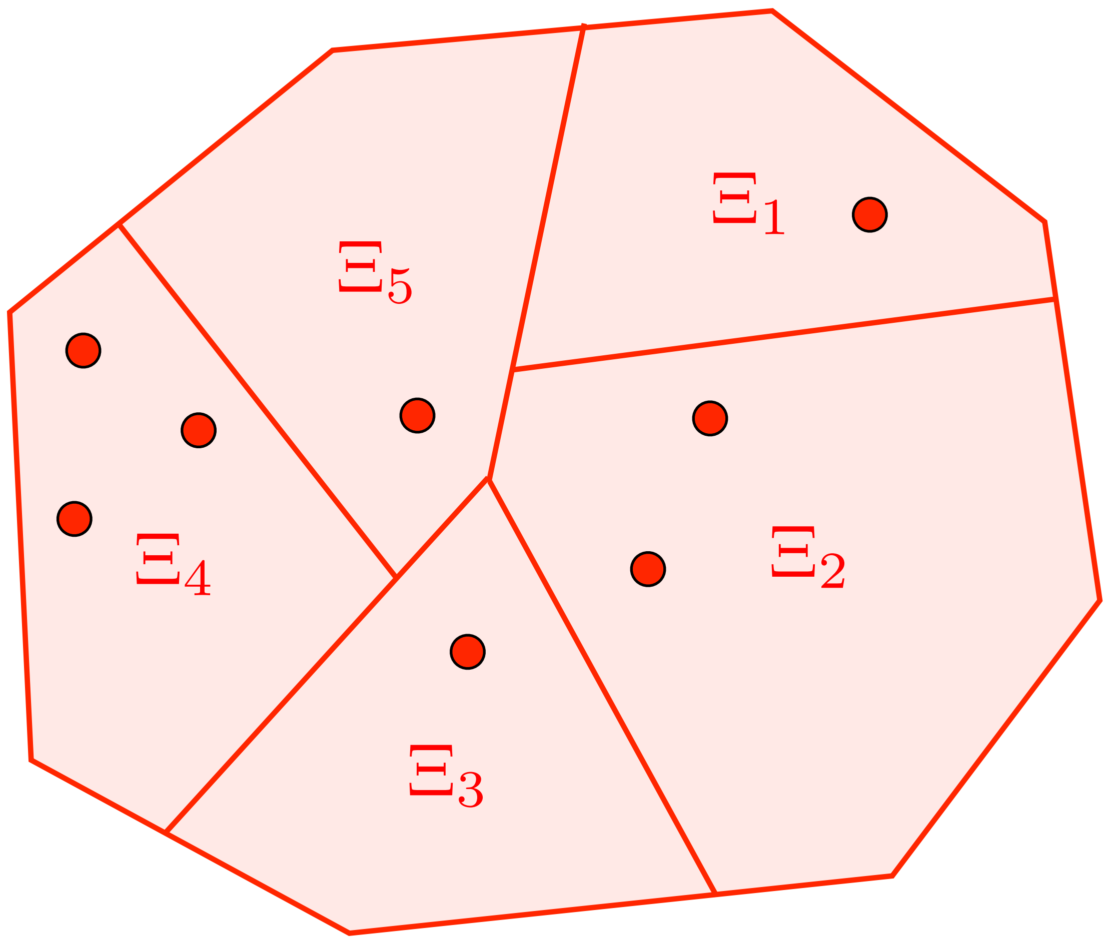
Part 4

Discrete Recourse Decisions

- Decision Rules
- **Iterative Partitioning**
- *K*-Adaptability

Bertsimas and Dunning. *Multistage Robust Mixed-Integer Optimization with Adaptive Partitions*. OR, 2016.
Postek and den Hertog. *Multistage Adjustable Robust Mixed-Integer Optimization via Iterative Splitting of the Uncertainty Set*. INFORMS JoC, 2016.

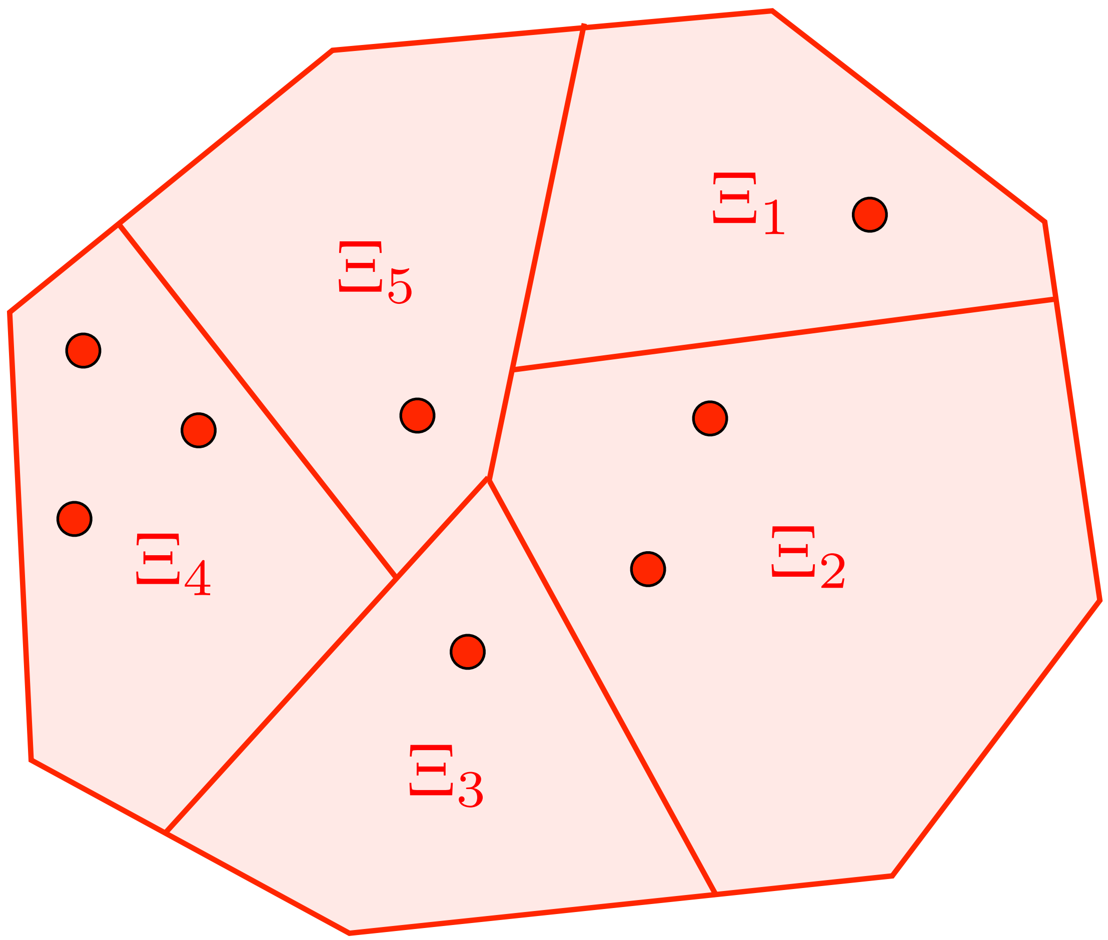
Main challenge with iterative and Voronoi partitioning:



dots = scenarios that are binding for at least one of the constraints in the problem

active (or critical) scenarios

Main challenge with iterative and Voronoi partitioning:

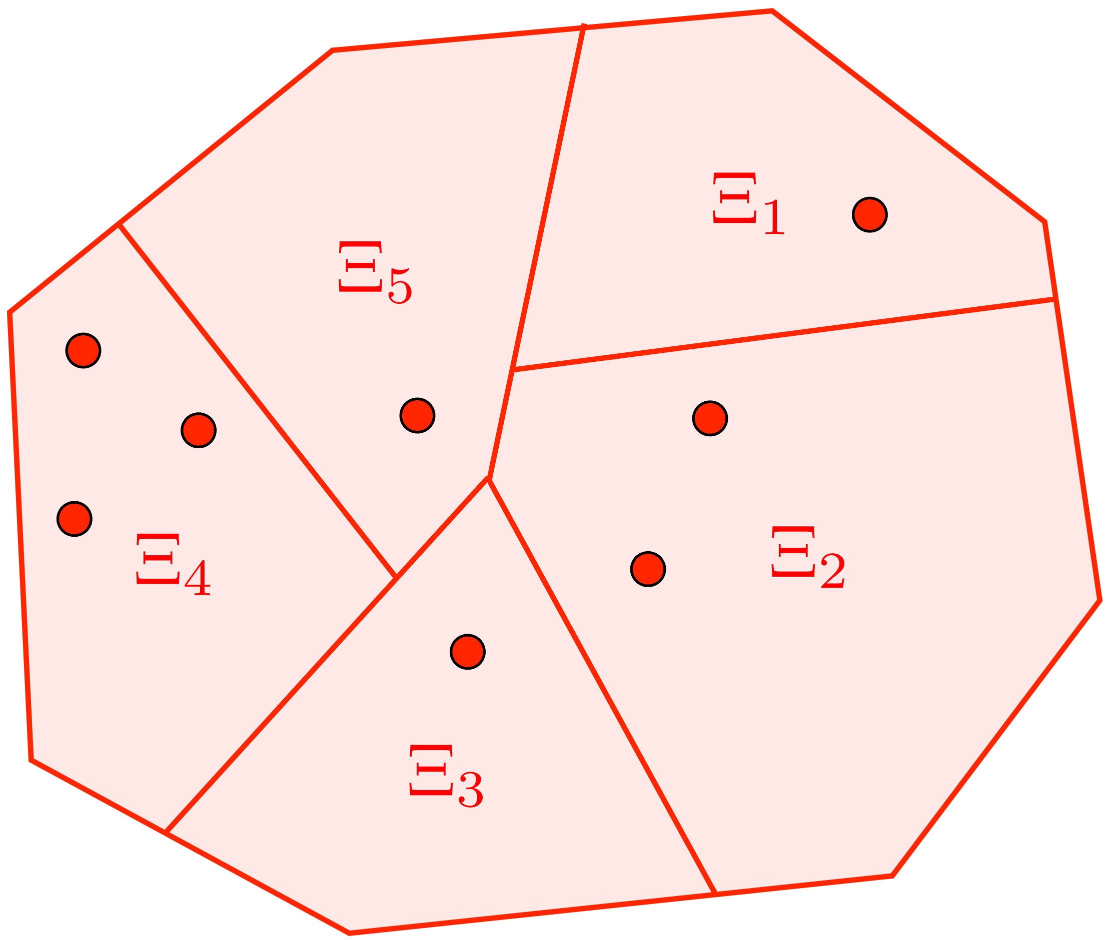


dots = scenarios that are binding for at least one of the constraints in the problem

active (or critical) scenarios

With discrete decisions, active scenarios may correspond to non-binding constraints:

Main challenge with iterative and Voronoi partitioning:



dots = scenarios that are binding for at least one of the constraints in the problem

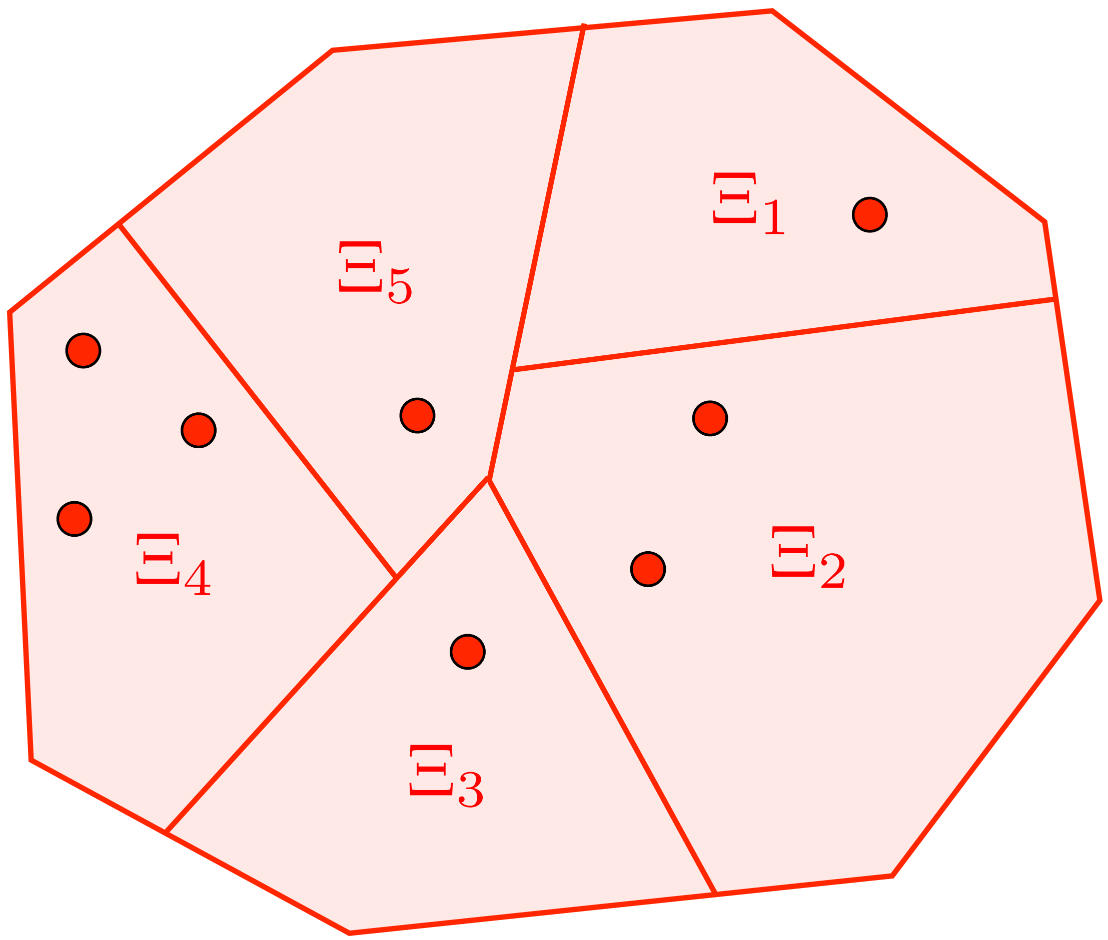
active (or critical) scenarios

With discrete decisions, active scenarios may correspond to non-binding constraints:

$$y_i \leq \xi_i \qquad \forall \xi_i \in [1/2,\ 1]$$
$$y_i \in \{0, 1\}$$

any $\xi_i^\star < 1$ is an active scenario!

Main challenge with iterative and Voronoi partitioning:



dots = scenarios that are binding for at least one of the constraints in the problem

active (or critical) scenarios

Can try to heuristically identify active scenarios:

$$\boldsymbol{\xi}^\star \in \arg\min_{\boldsymbol{\xi} \in \Xi} \left\{ [\boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x}^\star + \boldsymbol{W}\boldsymbol{y}^\star(\boldsymbol{\xi})]_i - \boldsymbol{h}_i(\boldsymbol{\xi}) \right\}$$

Part 4 **Discrete Recourse Decisions**

- Decision Rules
- Iterative Partitioning
- ***K*-Adaptability**

Bertsimas and Caramanis. *Finite Adaptability in Multistage Linear Optimization*. IEEE TAC, 2010.
Hanasusanto, Kuhn and W. *K-Adaptability in Two-Stage Robust Binary Programming*. OR, 2015.
Subramanyam, Gounaris and W. *K-Adaptability in Two-Stage Mixed-Integer Robust Optimization*. MPC, 2020.

K-Adaptability: Motivation

Recall the two-stage robust optimization problem:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

K-Adaptability: Motivation

Decide on alternative recourse decisions here-and-now…

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boxed{\boldsymbol{y} \in \mathcal{Y}} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

$$\begin{array}{c} \boldsymbol{y}^1 \in \mathcal{Y} \\ \boldsymbol{y}^2 \in \mathcal{Y} \\ \vdots \\ \boldsymbol{y}^K \in \mathcal{Y} \end{array}$$

K-Adaptability: Motivation

…and pick the best among the feasible decisions wait-and-see:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boxed{\boldsymbol{y} \in \mathcal{Y}} \end{array}$$

$$\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})$$

$$\begin{array}{c} \boldsymbol{y}^1 \in \mathcal{Y} \\ \boldsymbol{y}^2 \in \mathcal{Y} \\ \vdots \\ \boldsymbol{y}^K \in \mathcal{Y} \end{array}$$

$$\min_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \; : \; \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\}$$

K-Adaptability: Motivation

…and pick the best among the feasible decisions wait-and-see:

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X} \end{array} \quad + \quad \begin{array}{ll} \max & \boldsymbol{q}(\boldsymbol{\xi})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\xi} \in \Xi \end{array} \quad + \quad \cancel{\begin{array}{ll} \min & \boldsymbol{r}^\top \boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in \mathcal{Y} \end{array}}$$

$$\cancel{\boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y} \geq \boldsymbol{h}(\boldsymbol{\xi})}$$

$$\begin{array}{l} \boldsymbol{y}^1 \in \mathcal{Y} \\ \boldsymbol{y}^2 \in \mathcal{Y} \\ \vdots \\ \boldsymbol{y}^K \in \mathcal{Y} \end{array}$$

$$\min_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \;:\; \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\}$$

$$\begin{array}{ll} \underset{\boldsymbol{x}, \{\boldsymbol{y}^k\}_k}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max\limits_{\boldsymbol{\xi} \in \Xi} \min\limits_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \;:\; \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\} \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y},\ k \in \mathcal{K} \end{array}$$

K-Adaptability: Approximation Quality and Solvability

$$\begin{array}{ll} \underset{\boldsymbol{x},\{\boldsymbol{y}^k\}_k}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max_{\boldsymbol{\xi} \in \Xi} \min_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \, : \, \boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\} \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \, k \in \mathcal{K} \end{array}$$

K-Adaptability: Approximation Quality and Solvability

$$\begin{array}{ll} \underset{\boldsymbol{x}, \{\boldsymbol{y}^k\}_k}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max\limits_{\boldsymbol{\xi} \in \Xi} \min\limits_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \,:\, \boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\} \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y},\, k \in \mathcal{K} \end{array}$$

Objective Uncertainty

Constraint Uncertainty

K-Adaptability: Approximation Quality and Solvability

$$\begin{array}{ll} \underset{\boldsymbol{x}, \{\boldsymbol{y}^k\}_k}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max_{\boldsymbol{\xi} \in \Xi} \min_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \; : \; \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\} \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \; k \in \mathcal{K} \end{array}$$

	Objective Uncertainty	**Constraint Uncertainty**
 **Approximation Quality**	***optimal* whenever** $K > \min\{\dim \mathcal{Y}, \, \dim \Xi\}$	

K-Adaptability: Approximation Quality and Solvability

$$\begin{array}{ll} \underset{\boldsymbol{x}, \{\boldsymbol{y}^k\}_k}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max\limits_{\boldsymbol{\xi} \in \Xi} \min\limits_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \; : \; \boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\} \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \;\; \boldsymbol{y}^k \in \mathcal{Y}, \, k \in \mathcal{K} \end{array}$$

	Objective Uncertainty	**Constraint Uncertainty**
Approximation Quality	***optimal whenever*** $K > \min\{\dim \mathcal{Y}, \, \dim \Xi\}$	***can be suboptimal for all*** $K < \lvert\mathcal{Y}\rvert$

K-Adaptability: Approximation Quality and Solvability

$$\begin{aligned}\underset{\boldsymbol{x},\{\boldsymbol{y}^k\}_k}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \max_{\boldsymbol{\xi}\in\Xi} \min_{k\in\mathcal{K}} \left\{\boldsymbol{r}^\top \boldsymbol{y}^k \; : \; \boldsymbol{T}(\boldsymbol{\xi})\,\boldsymbol{x} + \boldsymbol{W}\boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi})\right\} \\ \text{subject to} \quad & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \; k \in \mathcal{K}\end{aligned}$$

	Objective Uncertainty	**Constraint Uncertainty**
Approximation Quality	***optimal* whenever** $K > \min\{\dim \mathcal{Y},\, \dim \Xi\}$	***can be suboptimal* for all** $K < \lvert\mathcal{Y}\rvert$
Solvability	***exact reformulation as polynomial-size MILP***	

K-Adaptability: Approximation Quality and Solvability

$$\begin{array}{ll} \underset{\boldsymbol{x}, \{\boldsymbol{y}^k\}_k}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max_{\boldsymbol{\xi} \in \Xi} \min_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \; : \; \boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\} \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \; k \in \mathcal{K} \end{array}$$

	Objective Uncertainty	**Constraint Uncertainty**
Approximation Quality	***optimal* whenever** $K > \min\{\dim \mathcal{Y}, \dim \Xi\}$	***can be suboptimal* for all** $K < \lvert\mathcal{Y}\rvert$
Solvability	***exact reformulation as polynomial-size MILP***	***exact reformulation as MILP:*** exponential in K polynomial in rest

K-Adaptability: Approximation Quality and Solvability

$$\begin{array}{ll} \underset{\boldsymbol{x}, \{\boldsymbol{y}^k\}_k}{\text{minimize}} & \boldsymbol{c}^\top \boldsymbol{x} + \max_{\boldsymbol{\xi} \in \Xi} \min_{k \in \mathcal{K}} \left\{ \boldsymbol{r}^\top \boldsymbol{y}^k \; : \; \boldsymbol{T}(\boldsymbol{\xi})\, \boldsymbol{x} + \boldsymbol{W} \boldsymbol{y}^k \geq \boldsymbol{h}(\boldsymbol{\xi}) \right\} \\ \text{subject to} & \boldsymbol{x} \in \mathcal{X}, \quad \boldsymbol{y}^k \in \mathcal{Y}, \; k \in \mathcal{K} \end{array}$$

	Objective Uncertainty	**Constraint Uncertainty**
Approximation Quality	***optimal* whenever** $K > \min\{\dim \mathcal{Y}, \dim \Xi\}$	***can be suboptimal* for all** $K < \lvert\mathcal{Y}\rvert$
Solvability	***exact reformulation as polynomial-size MILP***	***exact reformulation as MILP:*** exponential in K; polynomial in rest

<u>Solution</u> as monolithic MILP or via dedicated B&B schemes!

Part 5 Future Research Directions

- Limited-memory decision rules and their optimality (see, *e.g.*, Lu and Sturt, Operations Research, 2025)

- Limited-memory decision rules and their optimality (see, *e.g.*, Lu and Sturt, Operations Research, 2025)
- Continuous-time models

What the Future?

- Limited-memory decision rules and their optimality (see, *e.g.*, Lu and Sturt, Operations Research, 2025)
- Continuous-time models
- Decision-dependence
  - Time of information discovery depends on decisions only: *e.g.*, oil drilling
  - Time of information discovery depends on decisions and uncertainty: *e.g.* project scheduling

- Limited-memory decision rules and their optimality (see, *e.g.*, Lu and Sturt, Operations Research, 2025)
- Continuous-time models
- Decision-dependence
  - Time of information discovery depends on decisions only: *e.g.*, oil drilling
  - Time of information discovery depends on decisions and uncertainty: *e.g.* project scheduling
- Multi-stage (mixed-)integer & combinatorial problems (*e.g.*, total unimodularity?)

What the Future?

- Limited-memory decision rules and their optimality (see, *e.g.*, Lu and Sturt, Operations Research, 2025)
- Continuous-time models
- Decision-dependence
  - Time of information discovery depends on decisions only: *e.g.*, oil drilling
  - Time of information discovery depends on decisions and uncertainty: *e.g.* project scheduling
- Multi-stage (mixed-)integer & combinatorial problems (*e.g.*, total unimodularity?)
- Multi-stage distributionally robust optimization?

What the Future?

- Limited-memory decision rules and their optimality (see, *e.g.*, Lu and Sturt, Operations Research, 2025)
- Continuous-time models
- Decision-dependence
  - Time of information discovery depends on decisions only: *e.g.*, oil drilling
  - Time of information discovery depends on decisions and uncertainty: *e.g.* project scheduling
- Multi-stage (mixed-)integer & combinatorial problems (*e.g.*, total unimodularity?)
- Multi-stage distributionally robust optimization?
- Extensive numerical comparison of different approaches